Midterm #2 of ECE302, Prof. Wang's section
8–9pm Thursday, March 3, 2011, MTHW 210.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 12 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

Signature:

Question 1: [25%, Work-out question, Outcome 1]

1. [10%] X is a binomial random variable with $n = 3$ and $p = 1/3$. Find the conditional expectation $E(X|X \geq 1)$.

2. [15%] X is an exponential random variable with $\lambda = 1$. Find the probability $P(\sin(X \cdot \pi) > 0)$. Hint: when $X = 22.5$, $\sin(X \cdot \pi) = 1$, and when $X = 9.5$, $\sin(X \cdot \pi) = -1$.

Question 2: [10%, Work-out question, Outcome 1] X is a Bernoulli random variable with $p = \frac{1}{\sqrt{1.7}}$. Construct a new random variable $Y = X^{1.7}\ln(1+X)$. Find the variance $\mathsf{Var}(Y)$.

Question 3: [30%, Work-out question, Outcome 1] Consider a continuous random variable X with probability density function (pdf) being

$$f_X(x) = \begin{cases} \frac{x+2}{4} & \text{if } -2 \leq x < 0 \\ \frac{2-x}{4} & \text{if } 0 \leq x < 2 \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

Consider a new random variable $Y = \min(1, \max(-1, X))$.

1. [5%] Plot Y versus X for the range of $X = -2$ to 2. (Hint: What are the values of Y when $X = -2$? Repeat this question for $X = -1$, 0, 1, 2, respectively.)

2. [15%] Find the cumulative distribution function (cdf) of Y.

3. [10%] Find the pdf of Y. If you do not know the answer to the previous question, you can assume the cdf is

$$F_Y(y) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} - \frac{1}{2}e^{-x} & \text{if } 0 \leq x < 2 \\ 1 - \frac{1}{2}e^{-x} & \text{if } 2 \leq x. \end{cases} \tag{2}$$

Question 4: [10%, Work-out question, Outcome 1] In average, there are 1.5 new customers per second who arrive in an Apple store to purchase the new iPads. (Each customer is limited to one purchase.) Suppose the actual number of customers is a Poisson random variable.

An Apple store has 500 iPads in stock. What is the probability that the iPads are sold out in 5 minutes? Note: You solution will be in terms of summation (or integration). There is no need to expand/compute the summation (or the integration).

Question 5: [25%, Work-out question, Outcome 1] Consider a discrete random variable X with the following probability mass function:

$$p_k = \begin{cases} \frac{1}{6}\left(\frac{2}{3}\right)^{k-4} & \text{if } k \geq 4 \\ \frac{1}{2} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

1. [15%] Find the characteristic function $\Phi_X(\omega)$ of the above X.
2. [10%] Use the moment theorem to find $E(X)$.

   If you do not know the answer to the previous question, you can assume $\Phi_X(\omega) = \frac{2e^{j4\omega}}{3} + \frac{1}{3-j\omega}$ and solve $E(X)$ by the moment theorem.

   If you use other method to find $E(X)$, you will receive only 7 points.

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \leq p \leq 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega} - 1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\text{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\text{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\text{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$