Midterm #1 of ECE302, Prof. Wang's section
6:30-7:30pm Tuesday, February 1, 2011, MTHW 210.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 14 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [10%, Work-out question, Outcome 1] Consider a function $f(n) = e^{-2|n|}$. Find the values of

$$\sum_{n=-\infty}^{\infty} \max(1, n) f(n). \tag{1}$$

$$= \sum_{n=-\infty}^{0} (1) e^{-2|n|} + \sum_{n=1}^{\infty} n e^{-2|n|} \qquad \text{Let } k = -n$$

$$= \sum_{k=0}^{\infty} e^{-2k} + \sum_{n=1}^{\infty} n e^{-2n}$$

$$= \sum_{k=0}^{\infty} (e^{-2})^{k} + \sum_{n=1}^{\infty} n (e^{-2})^{n}$$

$$= \frac{1}{1-e^{-2}} + \frac{e^{-2}}{(1-e^{-2})^{2}}$$

Question 2: [15%, Work-out question, Outcome 1] Consider a function

$$f(n_1, n_2) = \begin{cases} n_2 2^{-n_1} & \text{if } 0 \le n_2 \le n_1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Find the value of

$$\sum_{n_1=-\infty}^{20} \sum_{n_2=-\infty}^{2} f(n_1, n_2). \tag{3}$$

$$= \sum_{n_1=0}^{20} \sum_{n_2=0}^{\min(n_1,2)} n_2 2^{-n_1}$$

$$= \sum_{n_1=2}^{20} \sum_{n_2=0}^{2} n_2 2^{-n_1} + \sum_{n_1=0}^{0} \sum_{n_2=0}^{0} n_2 2^{-n_1} + \sum_{n_1=1}^{1} \sum_{n_2=0}^{1} n_2 2^{-n_1}$$

$$= \left(\sum_{n_1=2}^{20} 2^{-n_1}\right)\left(\sum_{n_2=0}^{2} n_2\right) + 0 + (1)(2^{-1})$$

Let $k = n_1 - 2$
$n_1 = k+2$

$$= \left(\sum_{k=0}^{20} 2^{-k-2}\right)(0+1+2) + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{3}{4} \sum_{k=0}^{20} \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{2} + \frac{3}{4} \frac{1 - \left(\frac{1}{2}\right)^{21}}{1 - \frac{1}{2}} = \frac{1}{2} + \frac{3}{4}(2)\left(1 - \left(\frac{1}{2}\right)^{21}\right)$$

$$= 2 - \frac{3}{2}\left(\frac{1}{2}\right)^{21}$$

Question 3: [10%, Work-out question, Outcome 1] Consider a function

$$f(r,\theta) = e^{-r^2}. \tag{4}$$

Compute the value of

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} f(r,\theta) r dr d\theta. \tag{5}$$

Hint: You may need to use the change of variable formula: $s = r^2$.

$$\int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr\, d\theta \qquad u = r^2 \qquad du = 2r dr$$

$$r dr = \frac{1}{2} du$$

$$= \int_0^{2\pi} \int_0^{\infty} \frac{1}{2} e^{-u} du\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2} e^{-u} \Big|_0^{\infty} d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} (0 - 1) d\theta$$

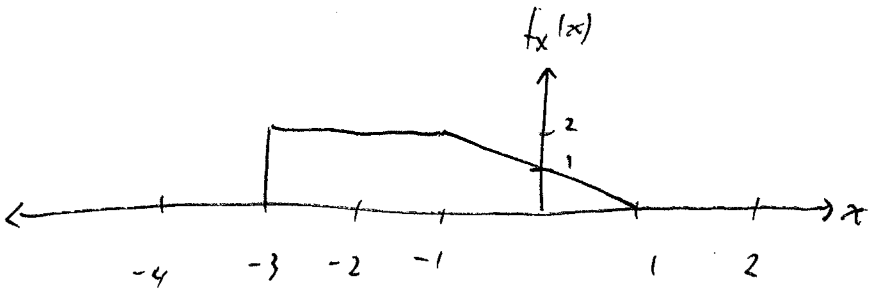
$$= \frac{1}{2} \int_0^{2\pi} d\theta = \frac{1}{2}(2\pi) = \pi$$

Question 4: [15%, Work-out question, Outcome 1] Consider a function

$$f(x) = \begin{cases} 2 & \text{if } -3 \le x < -1 \\ 1 - x & \text{if } -1 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{6}$$

1. [3%] Plot $f(x)$ for the range of $-4 < x < 2$.
2. [12%] Construct a new function $F(x)$ from $f(x)$: $F(x) = \int_{s=-\infty}^{x} f(s)ds$. Find the complete expression of $F(x)$. Hint: You need to consider different ranges of x.

1.



2. for $x < -3$, $F_X(x) = 0$

for $-3 \le x < -1$, $F_X(x) = \int_{-3}^{x} (2)\, ds = 2s\Big|_{-3}^{x} = 2(x+3)$

for $-1 \le x < 1$, $F_X(x) = 4 + \int_{-1}^{x} (1-s)ds$

$= 4 + \left[s - \frac{1}{2}s^2\right]_{-1}^{x} = 4 + \left[(x+1) - \frac{1}{2}(x^2 - 1)\right]$

$= 4 + x + 1 - \frac{1}{2}x^2 + \frac{1}{2}$

$= \frac{11}{2} + x - \frac{1}{2}x^2$

for $x \ge 1$, $F_X(x) = \cancel{4}\ 6$

$$F_X(x) = \begin{cases} 0 & x < -3 \\ 2(x+3) & -3 \le x < -1 \\ \frac{11}{2} + x - \frac{1}{2}x^2 & -1 \le x < 1 \\ 6 & x \ge 1 \end{cases}$$

Question 5: [15%, Work-out question, Outcome 1] Consider the following experiment: We first flip a fair coin. If the result is "head," then we roll a 6-faced fair die. If the result is "tail," we roll an unfair 6-faced die that has the following probability distribution.

$$P(X=k)=\begin{cases} 1/21 & \text{k=1} \\ 2/21 & \text{k=2} \\ 3/21 & \text{k=3} \\ 4/21 & \text{k=4} \\ 5/21 & \text{k=5} \\ 6/21 & \text{k=6} \end{cases} \tag{7}$$

Answer the following questions:

1. [5%] What is the sample space of the above experiment? Assuming the outcomes of the coin and the dices are independent, what is the corresponding probability distribution?

2. [5%] What is the probability that the face value of the die (it can be the fair or the unfair one depending on the outcome of the coin) is a NOT a prime number?

3. [5%] Conditioning that the face value of the die is between 3 and 5, what is the conditional probability the outcome of the coin is "head"? That is, we would like to compute

$$P(\text{head}|\text{the value of the die satisfies } 3 \leq X \leq 5). \tag{8}$$

$\cancel{1.\ S=\{1,2,3,4,5,6\}}$

1. $S=\{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

$$P: \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{42} \quad \frac{2}{42} \quad \frac{3}{42} \quad \frac{4}{42} \quad \frac{5}{42} \quad \frac{6}{42}$$

2. $P(\text{not Prime}) = P((H,1)) + P((H,4)) + P((H,6)) + P((T,1)) + P((T,4)) + P((T,6))$

$$= \frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{42}+\frac{4}{42}+\frac{6}{42} = \frac{3}{12}+\frac{11}{42} = \frac{21+22}{84} = \frac{43}{84}$$

3. $P(H|3\leq X\leq 5) = \dfrac{P(H,\ 3\leq X\leq 5)}{P(3\leq X\leq 5)} = \dfrac{\frac{1}{12}+\frac{1}{12}+\frac{1}{12}}{\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{3}{42}+\frac{4}{42}+\frac{5}{42}} = \dfrac{\frac{3}{12}}{\frac{3}{12}+\frac{12}{42}} = \dfrac{\frac{1}{4}}{\frac{1}{4}+\frac{2}{7}}$

$$= \frac{\frac{1}{4}}{\frac{7+8}{28}} = \frac{28}{4(15)} = \frac{7}{15}$$

Question 6: [20%, Work-out question, Outcome 1] Consider the x-y plane. Consider a disk (circle) of radius 2 on the plane, which are those points satisfying $\sqrt{x^2+y^2} \leq 2$.

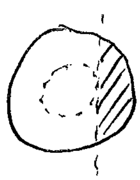
We throw a dart at the disk and we know that the dart will land uniformly likely on the disk. Let X and Y denote the x and y coordinates of the landing location of the dart. Answer the following questions:

1. [5%] Consider an event $A = \{X^2 + Y^2 \geq 1\}$: What is the probability of the event A?
2. [5%] Consider an event $B = \{X + Y > 0\}$: What is the probability of the event B?
3. [5%] Are events A and B independent?
4. [5%] Consider an event $C = \{X > 1\}$. Are events A and C independent?

1. $P(A) = 1 - \frac{\pi(1)^2}{\pi(2)^2} = 1 - \frac{1}{4} = \frac{3}{4}$  

2. $\frac{1}{2}$ 

3. $P(A \cap B) = (\frac{1}{2})(\frac{3}{4}) = P(A)P(B)$ Yes, A & B are independent 

4.  $P(A \cap C) = P(C) \neq P(A)P(C)$

No, A & C are NOT independent

Question 7: [15%, Work-out question, Outcome 1] Consider a continuous random variable X with its probability distribution specified by the following probability density function (pdf)

$$f_X(x) = \begin{cases} ce^{-3x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Answer the following questions:

1. What is the value of c so that $f_X(x)$ is a valid distribution?
2. What is the probability $P(\min(3, X) > 1)$?
3. What is the conditional probability $P(X > 5 | X > 7 \text{ or } X < 4)$?

1. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = c\int_0^{\infty} e^{-3x}\,dx = c\left(\frac{-1}{3}\right)e^{-3x}\Big|_0^{\infty} = \frac{-c}{3}(0-1) = \frac{c}{3}$$

$$\frac{c}{3} = 1 \quad \Rightarrow \quad c = 3$$

2. $$P(\min(X,3) > 1) = P(X > 1) = \int_1^{\infty} 3e^{-3x}\,dx$$
$$= -e^{-3x}\Big|_1^{\infty} = e^{-3}$$

3. $P(X>5 | X>7 \text{ or } X<4) = \cancel{P(X>5|X>7) + P(X>5|X<4)}$

$$= \frac{P(X>5 \text{ and } (X>7 \text{ or } X<4))}{P(X>7 \text{ or } X<4)}$$

$$P(X>7 \text{ or } X<4) = P(X>7) + P(X<4)$$
$$= e^{-(7)3} + 1 - e^{-(4)3} = 1 - e^{-12} + e^{-21}$$

$$P(X>5 \text{ and } (X>7 \text{ or } X<4)) = P(X>7) = e^{-21}$$

$$= \frac{e^{-21}}{1 - e^{-12} + e^{-21}}$$