Final Exam of ECE302, Prof. Wang's section

3:20–5:20pm Thursday, May 5, 2011, CL50 224.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. You have two hours to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 16 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution

Student ID:

E-mail:

Signature:

Question 1: [20%, Work-out question] Consider a random variable X. The probabilities $X = 1$ and $X = -1$ are $1/2$ and $1/2$, respectively. We are not able to directly observe X. Instead, we observe $Y = X + N$ where N is a random variable with probability density function

$$f_N(n) = 0.5e^{-|n|}. \tag{1}$$

X and N are independent.

1. [10%] Find out the linear MMSE estimator of X given $Y = y$. That is, your answer should be a function $\hat{X}_{\text{lin.MMSE}}(y)$.

   Note: If you do not know how to solve this problem, write down what MMSE stands for and you will receive 4 points.

2. [10%] Find out the MMSE estimator of X given $Y = y$. That is, your answer should be a function $\hat{X}_{\text{MMSE}}(y)$.

   Note: If you do not know how to solve this problem, answer which estimator has better performance: a MMSE estimator or a linear MMSE estimator. You will receive 2 points.

1. $m_X = 0$. $\quad m_Y = 0 + 0 = 0$

$\sigma_X^2 = 1 \quad \sigma_Y^2 = 1 + \sigma_N^2 = 3$

$\sigma_N^2 = E(N^2) = 1 + 1 = 2$

(By looking up the R.V table w. exponential distri.)

$$Cov(X,Y) = E(XY) = E(X^2) + E(XN) = 1$$

$$\hat{X}_{\text{LIN.MMSE}}(y) = \rho \frac{\sigma_X}{\sigma_Y}(y - m_Y) + m_X$$

$$= \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \cdot y = \frac{1}{3} y$$

2. $$P(X=1 \mid Y=y) = \begin{cases} \dfrac{0.5e^{-(1-y)}}{0.5e^{-(1-y)} + 0.5e^{-(-1-y)}} = \dfrac{e^{-1}}{e^{-1}+e^{1}} & \text{if } y < -1 \end{cases}$$

$$\begin{cases} \dfrac{0.5e^{-(1-y)}}{0.5e^{-(1-y)}+0.5e^{-(y+1)}} & \text{if } -1<y<1 \\ = \dfrac{e^{y-1}}{e^{y-1}+e^{-1-y}} = \dfrac{e^{y}}{e^{y}+e^{-y}} & \\ \dfrac{0.5e^{-(y-1)}}{0.5e^{-(y-1)}+0.5e^{-(y+1)}} & \text{if } 1<y \\ = \dfrac{e^{1}}{e^{1}+e^{-1}} & \end{cases}$$

$\Rightarrow$ MMSE estimator

$= E(X \mid Y=y)$

$$= \begin{cases} \dfrac{e^{-1}-e^{1}}{e^{-1}+e^{1}} & \text{if } -1>y \\ \dfrac{e^{y}}{e^{y}+e^{-y}} & \text{if } -1<y<1 \\ \dfrac{e^{1}-e^{-1}}{e^{-1}+e^{1}} & \text{if } 1<y \end{cases}$$

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Question 2: [10%, Work-out question] X_1, X_2, and X_3 are three independent standard Gaussian random variable. Construct three new random variables by

$$Y_1 = X_1 + X_2 + X_3 \tag{2}$$
$$Y_2 = 2X_1 - X_2 - X_3 \tag{3}$$
$$Y_3 = X_2 - X_3 \tag{4}$$

1. [4%] Find the mean vector and the covariance matrix of (Y_1, Y_2, Y_3).
2. [4%] Write down the joint pdf of (Y_1, Y_2, Y_3).

   Note: If you do not know the answer to this question, write down what type of random is the (marginal) random variable Y_1. You will receive 2 point.
3. [2%] Write down the condition pdf Y_2 given $Y_1 = 5$ and $Y_3 = 2$.

1. $m_{Y_1} = 0+0+0 = 0.$ $\quad \vec{m} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$m_{Y_2} = 2\times 0 - 0 - 0 = 0.$

$m_{Y_3} = 0 - 0 = 0.$

$\sigma_{Y_1}^2 = 1+1+1 = 3.$

$\sigma_{Y_2}^2 = 4+1+1 = 6$

$\sigma_{Y_3}^2 = 1+1 = 2$

$Cov(Y_1, Y_2) = 2\times 1 + 0 + 0 + (-1)\times 1 + 0 + 0$
$+ 0 + 0 + (-1)\times 1 = 0.$

$Cov(Y_2, Y_3) = 0+0+0+ -1 + 0 + 0 + 1 = 0$

$Cov(Y_1, Y_3) = 0+0+1+0+0-1 = 0.$

$$K = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

2. They are indep.

$$\Rightarrow f_{Y_1 Y_2 Y_3}(y_1, y_2, y_3)$$

$$= \frac{1}{\sqrt{2\pi \times 3}} e^{-\frac{y_1^2}{2\times 3}} \cdot \frac{1}{\sqrt{2\pi 6}} e^{-\frac{y_2^2}{2\times 6}} \cdot \frac{1}{\sqrt{2\pi 2}} \times e^{-\frac{y_3^2}{2\times 2}}$$

3. They are indep.

$$f_{Y_2 | Y_1 Y_3}(y_2 | y_1 y_3)$$

$$= \frac{1}{\sqrt{2\pi 6}} e^{-\frac{y_2^2}{2\times 6}} \quad \#$$

Question 3: [5%, Work-out question]

1. [4%] X is a Poisson random variable with parameter $\alpha = 3$. Y is a Poisson random variable with parameter $\alpha = 5$. X and Y are independent. Let $Z = X + Y$. Write down the probability mass function p_k of the Z random variable.

   Also write down the characteristic function of Z.

   Note: If you do not know the answer to the above question, you can instead assume that both X and Y are independent binomial random variables with parameter $n = 2, p = 1/3$. You can still get 3 point.

2. [1%] What does the acronym "i.i.d." stand for?

1. Z is also a Poisson w. para

$$\alpha = 3 + 5 = 8$$

$$P_k = \frac{\alpha^k}{k!} e^{-\alpha}$$

$$= \frac{8^k}{k!} \cdot e^{-8}$$

$$\Phi_Z(\omega) = e^{8(e^{j\omega} - 1)}$$

\#

2. Independently & identically distributed.

Question 4: [10%, Work-out question] Demographically, 10% of the total population of West Lafayette are Purdue students. To conduct an opinion poll, a statistician randomly chooses 2500 residents of West Lafayette. What is the (approximate) probability that more than 270 (out of the total 2500 samples) are Purdue students?

Note: You may need to use the facts that $Q(1) = 0.1587$, $Q(4/3) = 0.0912$, $Q(5/3) = 0.0478$, and $Q(2) = 0.0228$.

$S = X_1 + \cdots + X_{2500}$

$E(X_i) = 0.1$

$Var(X_i) = 0.1 \times (1 - 0.1) = 0.09$.

$E(S) = 0.1 \times 2500 = 250$.

$Var(S) = 0.09 \times 2500 = 9 \times 25$

$\sigma_S = \sqrt{9 \times 25} = 15$.

$P(S > 270)$

$= P\left(\frac{S - 250}{15} > \frac{270 - 250}{15}\right)$

$= P\left(Z_{2500} > \frac{20}{15}\right)$

$\approx P(Z > \frac{4}{3}) = Q(\frac{4}{3}) = 0.0912$ #

Question 5: [15%, Work-out question] A random process $X(t)$ can be described by $X(t) = \cos(\pi(t+\Theta))$, where Θ is a continuous random variable that is uniformly distributed on the $(0,1)$ interval.

1. [7%] What is the probability $X(0.5) > 0$?
2. [6%] What is the correlation between $X(0.5)$ and $X(1)$?

   Note: You may need to use the following trigonometric formulas:

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \tag{5}$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \tag{6}$$

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \tag{7}$$

3. [2%] What is the definition of "correlation coefficient"?

1. $P(X(0,5) > 0)$

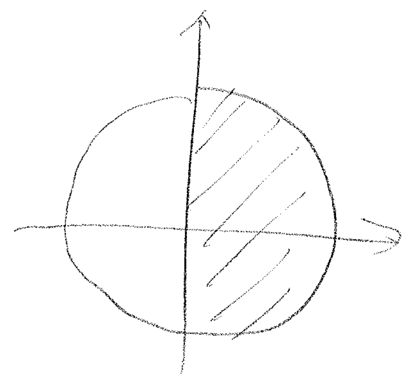
$= P(\cos(\pi(0,5+\Theta)) > 0)$

$= P(0 < \pi(0,5+\Theta) < 0,5\pi)$

$+ P(1,5\pi < \pi \times (0,5+\Theta) < 2\pi)$

$= P(-0,5 < \Theta < 0) + P(1 < \Theta < 1,5)$

$= 0 + 0 = 0.$



2. $E(\cos(\pi \times (0,5+\Theta)) \cdot \cos(\pi(1+\Theta)))$

$$= E\left(\frac{\cos(\pi(1.5+2\Theta)) + \cos(\pi(0.5))}{2} \right)$$

$$= \frac{E(\cos(\pi(1.5+2\Theta))) + E(\cos(0.5\pi))}{2}$$

$$= \frac{0+0}{2} = 0.$$

3. $$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$ #

Question 6: [10%, Work-out question] X is a Bernoulli random variable with parameter $p = 0.3$. Conditioning on $X = x_0$, Y is uniformly randomly distributed on the interval $(0, 1 + x_0)$. Find the probability $P(X + Y < 1)$.

$$P(X+Y<1)$$

$$= P(X=0,\ Y<1) + P(X=1,\ Y=0)$$

$$= 0.7 \times \frac{1}{1} + 0.3 \times 0$$

$$= 0.7 \ \#$$

Question 7: [10%, Work-out question] $p = \frac{1}{4}$

1. [5%] X is ~~an exponential~~ geometric random variable with $\cancel{\lambda}$ $p = 1/4$. Find the conditional expectation $E(X|X \le 2)$.

2. [5%] X is a binomial random variable with $n = 2$ and $p = 1/3$. Plot the corresponding cdf $F_X(x)$ for the range of $x = -1$ to 3. Please carefully mark the solid and empty end points of your piece-wise curve.

1. $P(X=0) = p(1-p)^0 = \frac{1}{4}.$

$P(X=1) = p(1-p)^1 = \frac{1}{4} \times \frac{3}{4}$

$P(X=2) = p(1-p)^2 = \frac{1}{4} \times \left(\frac{3}{4}\right)^2$

Conditional ~~pdf~~ expectation

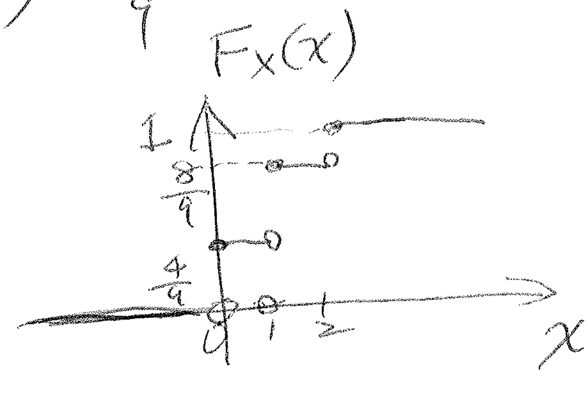
$$\frac{\frac{1}{4} \times 0 + \frac{1}{4} \times \frac{3}{4} \times 1 + \frac{1}{4} \times \left(\frac{3}{4}\right)^2 \times 2}{\frac{1}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \left(\frac{3}{4}\right)^2}$$

$$= \frac{12 + 18}{\cancel{64 + 12}\ 16 + 12 + 9} = \frac{30}{37} \#$$

2. $P(X=0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$P(X=1) = 2 \times \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right) = \frac{4}{9}$

$P(X=2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$



Question 8: [20%, Multiple choice question. There is no need to justify your answers]

N 1. [3%] X and Y are uniformly distributed in a unit square $(0,1) \times (0,1)$. We know that $Z = X + Y$ and $W = X - Y$. Are Z and W independent?

Y 2. [3%] X and Y are independent Gaussian random variables with $m_X = 1$, $\sigma_X^2 = 1$, $m_Y = -3$, and $\sigma_Y^2 = 1$. We know that $Z = X + Y$ and $W = X - Y$. Are Z and W independent?

N 3. [3%] X is a Gaussian random variable with $m_X = 0.1$, $\sigma_X^2 = 1$. Is the following statement correct? "By the Markov inequality, we must have $P(X \geq 1) \leq \frac{m_X}{1} = 0.1$."

N 4. [2%] Is the following statement true? "A 99% confidence interval is smaller than a 95% confidence interval."

N 5. [2%] Suppose we know that a random variable X_1 is independent of another random variable X_2, and X_2 is independent of X_3. Is the following statement true? "X_1 and X_3 must be independent."

N 6. [2%] Suppose we know that two random variables X_1 and X_2 are uncorrelated; and X_2 and X_3 are also uncorrelated. Is the following statement true? "X_1 and X_3 must be uncorrelated."

N 7. [2%] Is the following statement true? "Since a Maximum Likelihood (ML) detector *maximizes the likelihood function*, it outperforms the MAP detector."

Y 8. [3%] Is the following statement true? "A cumulative distribution function $F_X(x)$ is always non-decreasing and right continuous."