ECE 302, Midterm #3

8–9pm Thursday, April 9, 2009, EE 170.

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains only work-out questions. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 11 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.
6. You can rip off the table in the back of the exam booklet.

Name: Solution

Student ID:

E-mail:

Signature:

Question 1: [32%]

1. [9%] X is an exponential random variable with $\lambda = 2$. Y is a Laplacian random variable with $\alpha = 1$. What is the joint pdf of (X, Y)? Your answer should be of the following form

$$f_{X,Y}(x,y) = \begin{cases} \dots & \text{if } \dots \\ \dots & \text{if } \dots . \\ \dots & \text{if } \dots \end{cases} \qquad (1)$$

independent.

2. [6%] $Z = X + 2Y$. Find the mean and variance of Z.

3. [10%] Find $E(XYZ)$.

4. [7%] [Advanced] $W = \max(X, Y)$. Find the cdf of W.

1. $f_X(x) = 2e^{-2x}$, $x > 0$ ~~[illegible]~~ +4%

$f_Y(y) = \frac{1}{2}e^{-|y|}$, $-\infty < y < \infty$

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ ~~[illegible]~~ +5%

$$= \begin{cases} \frac{1}{2}e^{-|y|} \cdot 2e^{-2x} , & x \geq 0 \\ 0 , & x < 0 \end{cases}$$

2. $Z = X + 2Y$ ~~[illegible]~~ +3%

$E[Z] = E[X+2Y] = E[X] + 2E[Y]$

$= \frac{1}{2} + 2(0) = \frac{1}{2}$

$Var(Z) = E[(X+2Y)^2] - \frac{1}{2}^2$

$= E[X^2 + 4XY + 4Y^2] = E[X^2] + 4E[X]E[Y] + 4E[Y^2]$

+3%

$= Var(X) + E[X]^2 + 4(0) + 4Var(Y)$

$$\mathrm{Var}(X) = \frac{1}{4}$$

$$\mathrm{Var}(Y) = 2$$

$$\therefore \mathrm{Var}(Z) = \frac{1}{4} + \left(\frac{1}{2}\right)^2 + 4(2) - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4} + 8 = \boxed{\frac{33}{4}}$$

+3%

3. $E[XYZ] = E[XY(X+2Y)] = E[X^2Y + 2XY^2]$

$$= \cancel{E[X^2]E[Y]} + 2E[X]E[Y^2]$$

0 +4%

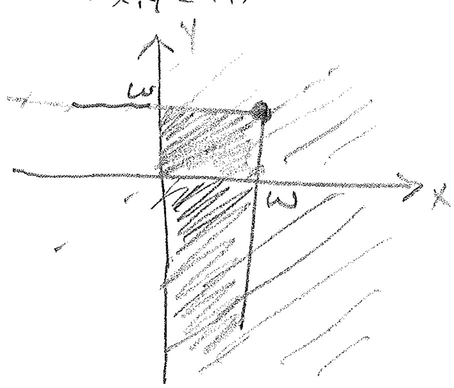
$$= 2 \cdot \frac{1}{2}(2) = \boxed{2}$$ +3%

4. $W = \max(X, Y)$

+4%

$$F_W(\omega) = \int_0^{\omega}\int_0^{\omega} f_{X,Y}(x,y)\,dx\,dy \quad , \omega \geq 0$$

$f_{X,Y}(x,y)$



$$= \int_{-\infty}^{\omega}\int_0^{\omega} \frac{1}{2}e^{-|y|}2e^{-2x}\,dx\,dy$$

$$= \int_{-\infty}^{\omega}\frac{1}{2}e^{-|y|}\int_0^{\omega} 2e^{-2x}\,dx\,dy$$

$$= \int_{-\infty}^{\omega}\frac{1}{2}e^{-|y|}\left[-e^{-2x}\right]_0^{\omega} dy$$

$$= \int_{-\infty}^{\omega}\frac{1}{2}e^{-|y|}\left[1 - e^{-2\omega}\right] dy$$

$$= (1 - e^{-2\omega})\int_{-\infty}^{\omega}\frac{1}{2}e^{-|y|}\,dy$$

+3%

$$= \begin{cases} (1 - e^{-2\omega})(1 - \frac{1}{2}e^{-\omega}) , & \omega \geq 0 \\ 0 , & \text{else} \end{cases}$$

Question 2: [27%] A game proceeds as follows. First toss a fair dice and use X to denote the outcome (the number of dots facing up). Once the value of X is decided, we flip a fair coin $2X$ times. (For example, if the outcome of the dice is 3, then we flip a fair coin 6 times.) Let Y denote the total number of heads (out of the $2X$ coin flips).

1. [3%] What is the marginal pmf $p_k = P(X = k)$.
2. [3%] What is the conditional pmf $p_{h|k} = P(Y = h|X = k)$.
3. [3%] What is the joint pmf $p_{k,h} = P(X = k, Y = h)$.
4. [5%] What is the marginal probability probability $P(Y = 0)$?
5. [4%] What is the conditional expectation $E(Y|X = x)$.
6. [4%] What is the expectation $E(Y)$?
7. [5%] [Advanced] What is the variance $\mathsf{Var}(Y)$?

1) $P_X(k) = \frac{1}{6}$, $k \in \{1, 2, \ldots, 6\}$ +3%

2) $P_{Y|X}(Y=h|X=k) = \binom{2k}{h}\left(\frac{1}{2}\right)^h\left(\frac{1}{2}\right)^{2k-h}$ +3%

3) $P_{Y|X}(y|x)\,P_X(x) = P_{Y,X}(y,x) = \frac{1}{6}\cdot\binom{2k}{h}\left(\frac{1}{2}\right)^h\left(\frac{1}{2}\right)^{2k-h}$ +3%

4) $P(Y=0) = \sum_{k=1}^{6} P(Y=0|X=k)\,P(X=k)$ +3%

$= 1/6 \cdot \left[\sum_{k=1}^{6}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{2k}\right] = \boxed{1/6 \sum_{k=1}^{6}\left(\frac{1}{4}\right)^k}$ +2%

5) $E[Y|X=x] = 2xp$ +4% $= 2(\frac{1}{2})x = x$

6) $E[Y] = E[E[Y|X]] = E[2xp] = 2p\,E[X] = 2p(\frac{7}{2})$ +2%
+2%
$= 7p = \frac{3}{2}$

7) $Var(Y) = E[Y^2] - E[Y]^2 = E[E[Y^2|X=x]] - (7p)^2$ +3%

$= E[2xp(1-p) + (2xp)^2] - (49p^2)$

$\frac{1}{6}[1^2+2^2$

$= 2p(1-p)E[X] + 4p^2 E[X^2] - 49p^2$

$= 7p(1-p) + 4p^2(Var(X) + E[X]^2) - 49p^2$

$= 7p(1-p) + 4p^2\left(\frac{35}{12} + \frac{49}{4}\right) - 49p^2$ +2%

$$= 7p(1-p) + 4p^2\left(\frac{182}{12}\right) - 49p^2$$

$$= 7p(1-p) + \frac{182}{3}p^2 - 49p^2$$

$$= \boxed{7p(1-p) + \frac{35}{3}p^2} \quad , \quad p = \frac{1}{2}$$

$$= \frac{7}{4} + \frac{35}{12} = \frac{56}{12} = \frac{28}{6} = \frac{14}{3}$$

Question 3: [20%] X is equally likely to take values in one of the three outcomes: $\{-1, 0, 1\}$. N is uniformly distributed on the interval $(-1, 1)$. X and N are independent. Let $Y = X + N$.

1. [15%] Find out the correlation coefficient between X and Y.
2. [5%] Are X and Y correlated or not? Orthogonal or not?

1. $\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$ +3%

$Var(X) = (-1)^2 \frac{1}{3} + (1)^2 \frac{1}{3} = \frac{2}{3}$ +3%

$E[X] = 0$

$E[N] = 0$

$Var(N) = \frac{(b-a)^2}{12} = \frac{1}{3}$

$Cov(X,Y) = E[XY] - E[X]E[Y]$ +3%

$= E[X(X+N)] - E[X]E[X+N]$

$= E[X^2 + XN] - E[X](E[X] + E[N])$

$= E[X^2] + E[X]E[N] - 0$

$= Var(X) = \frac{2}{3}$ +3%

$Var(Y) = Var(X) + Var(N) = 1$

$\therefore \rho_{XY} = \frac{\frac{2}{3}}{\sqrt{\frac{2}{3}} \cdot \sqrt{1}} = \boxed{\sqrt{\frac{2}{3}}}$ +3%

2. Yes, they are correlated. +3%
Not orthogonal. +2%

Question 4: [21%] Consider a Gaussian random variable X with $m = 3$ and $\sigma = 1$. Let $Y = -2X + 1$.

1. [6%] Write down the pdf of Y.
2. [10%] Find out the probability $P(2^{|Y|} > 2)$. Your answer should use the $Q(x)$ function where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.
3. [5%] [Advanced] Find out $E(\cos(Y))$. Hint: The Euler's formula.

1. $Y = -2X + 1, \quad X \sim \mathcal{N}(3, 1)$

$$E[Y] = E[-2X+1] = -2E[X] + 1 = -6 + 1 = -5 \quad +1\%$$

$$Var(Y) = Var(-2X+1) = 4Var(X) = 4 \quad +1\%$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{(y+5)^2}{2\cdot 4}} = \frac{1}{\sqrt{8\pi}} e^{-\frac{(y+5)^2}{8}} \quad +4\%$$

2. $$P(2^{|Y|} > 2) = P(\log_2(2^{|Y|}) > \log_2(2)) \quad +2\%$$

$$= P(|Y| > 1)$$

$$= P(Y \geq 1) + P(Y \leq -1) \quad +3\%$$

$$= Q\left(\frac{1-(-5)}{2}\right) + 1 - Q\left(\frac{-1-(-5)}{2}\right)$$

$$= Q(3) + 1 - Q(2) \quad +5\%$$

3. $$E[\cos(Y)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{8\pi}} e^{-\frac{(y+5)^2}{8}} \cos y \, dy \quad +2\%$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{8\pi}} e^{-(y+5)^2/8} e^{-jy} \, dy + \int_{-\infty}^{\infty} \frac{1}{\sqrt{8\pi}} e^{-(y+5)^2/8} e^{jy} \right]$$

$$= \frac{1}{2} \left[\left[\int_{-\infty}^{\infty} e^{j(-1)y} f_Y(y) \, dy \right] + \left[\int_{-\infty}^{\infty} e^{j(1)y} f_Y(y) \right] \right] + 2\%$$

$$= \frac{1}{2} \left[E\left[e^{j(-1)y}\right] + E\left[e^{j(1)y}\right] \right]$$

$$= \frac{1}{2} \left[\Phi_Y(-1) + \Phi_Y(1) \right] + 1\%$$

$$= \frac{1}{2} \left[e^{+j5 - \frac{4(-1)^2}{2}} + e^{-j5 - \frac{4(1)^2}{2}} \right]$$

$$= \boxed{\frac{1}{2} \left[e^{j5 - 2} + e^{-j5 - 2} \right]}$$