ECE 302, Midterm #2

6:30-7:30pm Thursday, March 5, 2009, EE 170,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains only work-out questions. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 12 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.
6. You can rip off the table in the back of the exam booklet.

Name: Solution

Student ID:

E-mail:

Signature:

Question 1: [20%]

1. [3%] X is an exponential random variable with $\lambda = 2$. Find $E((X - 0.5)^2)$
2. [3%] X is a binomial random variable with $n = 100$, $p = 0.1$. Find $P(X = 0|X < 2)$.
3. [3%] X is a Poisson random variable with $\alpha = 1$. $f(x)$ is a function such that

$$f(x) = \begin{cases} 3 & \text{if } -0.1 < x \leq 1.5 \\ -2 & \text{if } 1.5 < x \leq 1.9 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

   Find $E(f(X)^2)$.

4. [5%] X is a geometric random variable with $p = 1/5$. Find $E(2^X - X)$.
5. [6%] X is a uniform random variable with $a = 1$, $b = 3$. Find $E(\max(X, 2))$

1) $\lambda = 2 \quad E[X] = \frac{1}{\lambda} = 0.5$

$\therefore E[(X-0.5)^2] = E[(X-E[X])^2] = Var(X) = \frac{1}{2^2} = \boxed{0.25}$

2) $\binom{n}{k} p^k (1-p)^{n-k}$; $P(X=0 \mid X<2) = \dfrac{P(X=0 \cap X<2)}{P(X<2)}$

$= \dfrac{\binom{100}{0} p^0 (1-p)^{100}}{\sum_{k=0}^{1} \binom{100}{k} p^k (1-p)^{100-k}}$

$= \dfrac{(1-p)^{100}}{(1-p)^{100} + 100p(1-p)^{99}}$

$= \dfrac{(.9)^{100}}{(.9)^{100} + 10(.9)^{99}}$

$= \dfrac{.9}{.9 + 10} = \dfrac{.9}{10.9}$

3) $p_X(k) = \frac{\alpha^k e^{-\alpha}}{k!} = \frac{e^{-1}}{k!}$

$$E[f(X)^2] = \underbrace{(3)^2 \frac{e^{-1}}{0!}}_{(k=0)} + \underbrace{(3)^2 \frac{e^{-1}}{1!}}_{(k=1)}$$

$$= \boxed{18e^{-1}}$$

4) $p_X(k) = p(1-p)^k$, $p = 0.2$

$$E[2^X - X] = E[2^X] - E[X]$$

$$= E[2^X] - \frac{(1-p)}{p}$$

$$E[2^X] = \sum_{k=0}^{\infty} 2^k p(1-p)^k$$

$$= p\sum_{k=0}^{\infty}(2(1-p))^k = p\sum_{k=0}^{\infty}(2 \cdot 0.2)^k$$

$$= p\sum_{k=0}^{\infty} 0.4^k = \frac{0.8}{1-0.4} = 4/3$$

$$\therefore E[2^X - X] = 4/3 - \frac{0.2}{0.8} = \boxed{13/12}$$

$\cancel{5)}$ (not on exam) $f_X(x) = \frac{1}{2}$, $1 < x < 3$ $\qquad \max(X,2) = \begin{cases} x & , x > 2 \\ 2 & , x < 2 \end{cases}$

$$E[\max(X,2)] = \int_1^3 \max(x,2) f_X(x)\,dx$$

$$= \int_1^2 2\left(\tfrac{1}{2}\right)dx + \int_2^3 x \tfrac{1}{2}\,dx$$

$$= 1 + \left[\tfrac{1}{4}x^2\right]_2^3 = 1 + \tfrac{9}{4} - 1 = \boxed{\tfrac{9}{4}}$$

Question 2: [18%] Let X be a Bernoulli random variable with $p = 1/3$. Let Y be a binomial random variable with $n = 2$ and $p = 1/4$. Suppose X and Y are independent. Answer the following questions:

1. [3%] What is the sample space when considering jointly (X, Y)? What is the corresponding weight assignment?
2. [10%] Let $Z = X^3 Y$. Find and plot the cdf $F_Z(z)$.
3. [5%] Find and plot the generalized pdf $f_Z(z)$. If you do not know the answer to the previous question, you can assume $F_Z(z)$ being as follows.

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 0.25\sin(z) & \text{if } 0 \le z < \frac{\pi}{2} \\ \frac{z}{\pi} & \text{if } \frac{\pi}{2} \le z < \pi \\ 1 & \text{if } \pi \le z \end{cases} \tag{2}$$

1. $S_{XY} = \{(x,y) : x \in \{0,1\},\ y \in \{0,1,2\}\}$

$p(X=0) = 1-p = 2/3 \qquad p(X=1) = 1/3$

$p(Y=0) = \binom{2}{0} p^0 (1-p)^{2-0} = (3/4)^2 = 9/16 \qquad P(Y=1) = \binom{2}{1} p^1 (1-p)^1 = 2(1/4)(3/4) = 6/16$

$P(Y=2) = \binom{2}{2} p^2 (1-p)^0 = 1/16$

$(0,0) = \frac{2}{3} \cdot \frac{9}{16} = 3/8$

$(0,1) = \frac{2}{3} \cdot \frac{6}{16} = 1/4$

$\vdots$

X \ Y	0	1	2
0	3/8	1/4	1/24
1	3/16	1/8	1/48

2) $Z = X^3 Y$

$S_Z = \{0, 1, 2\}$

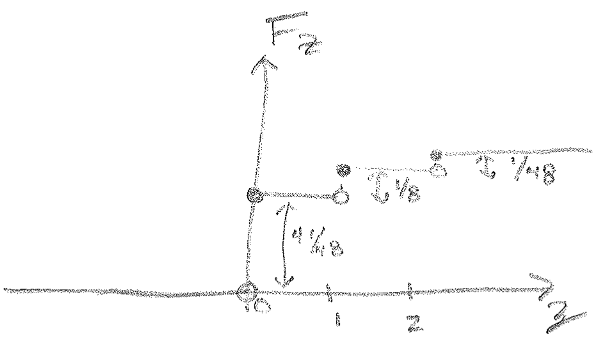
$P(Z=0) = P(\{(0,0), (0,1), (0,2), (1,0)\})$

$= 3/8 + 1/4 + 1/24 + 3/16 = 41/48$

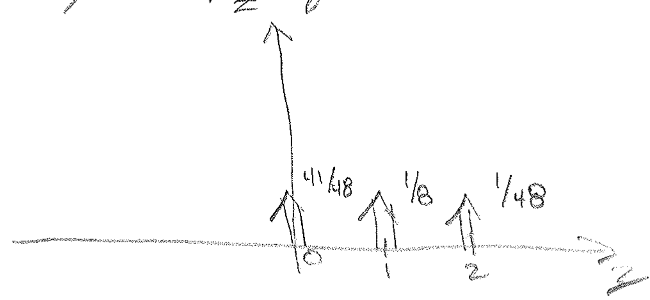
$P(Z=1) = P(\{(1,1)\}) = 1/8$

$P(Z=2) = P(\{(1,2)\}) = 1/48$

$F_Z(z) = P(Z \leq z)$

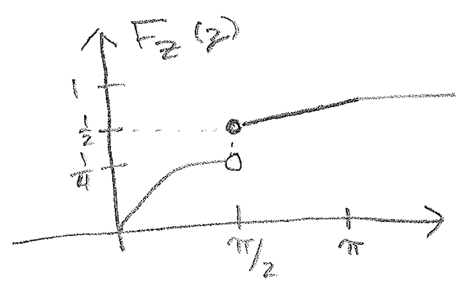


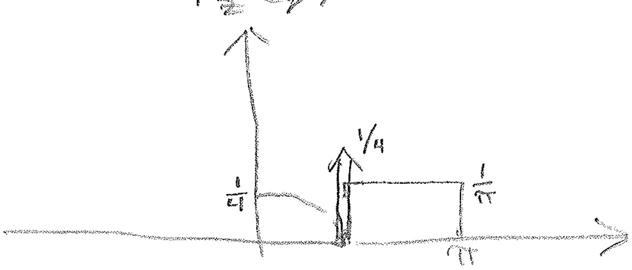
3) $f_Z(z)$



$f_Z(z)$ (Alternate):

$$f_Z(z) = \begin{cases} 0, & z < 0 \\ 0.25\cos(z), & 0 \leq z < \pi/2 \\ 1/\pi + 1/4\,\delta(z - \pi/2), & \pi/2 \leq z < \pi \\ 0, & \pi \leq z \end{cases}$$

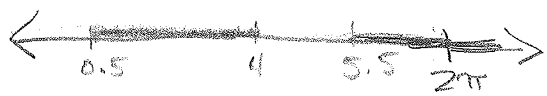




Question 3: [7%] Consider a discrete random variable W with cdf $F_W(w)$ being

$$F_W(w) = \begin{cases} 0 & \text{if } w < 0.5 \\ 0.01w^2 & \text{if } 0.5 \le w < 2\pi \\ 1 - \sin(\frac{1}{w^2}) & \text{if } 2\pi \le w \end{cases} \quad (6.28) \tag{3}$$

Find the probability $P(0.5 \le W < 4 \text{ or } 5.5 < W)$.



$= P(\{0.5 \le W < 4\} \cup \{5.5 < W\})$

$= P(0.5 \le W < 4) + P(W > 5.5)$

$= \boxed{F_W(4) - F_W(0.5) + (1 - F_W(5.5))}$

$= \boxed{(0.01)(16) - (0.01)(0.25) + (1 - (0.01)(5.5^2))}$

$= 0.16 - 0.0025 + (1 - (0.01)30.25)$

$= 0.16 - 0.0025 + (1 - 0.3025)$

$= 0.1575 + (0.6975)$

$= 0.8550$

Question 4: [25%] Consider a random variable X such that $S = (1, \infty)$, i.e., X can be any larger-than-one real numbers (say 1.1, π, etc.). We also know that the probability density of X is $f_X(x) = ce^{-2x}$ for $x > 1$ where c is a constant.

1. [3%] Find the c value.
2. [10%] Find the characteristic function $\Phi_X(\omega)$.
3. [12%] Use the moment theorem to find the mean and the variance of X. (If you do not know the answer of the previous question, you may assume $\Phi_X(\omega) = e^{j\omega - 1}(1 + \frac{1}{1-j\omega})$.)

1) $\int_1^\infty ce^{-2x}\,dx = -\frac{1}{2}c\, e^{-2x}\Big|_1^\infty = (0 + \frac{1}{2}c\, e^{-2}) = 1$

$c = \boxed{2e^2}$

2) $\Phi_X(\omega) = E[e^{j\omega X}]$

$= \int_1^\infty 2e^2 e^{-2x} e^{j\omega x}\,dx = 2e^2 \int_1^\infty e^{-(2-j\omega)x}\,dx$

$= \frac{-2e^2}{2-j\omega}\left[e^{-(2-j\omega)x}\right]_1^\infty$

$= \frac{-2e^2}{2-j\omega}\left(0 - e^{-(2-j\omega)}\right)$

$= \frac{2e^2(e^{-(2-j\omega)})}{2-j\omega}$

3)

$= \boxed{\frac{2e^{j\omega}}{2-j\omega}}$

$E[X] = \frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega)\Big|_{\omega=0} = \frac{1}{j}\left[\frac{d}{d\omega}\left[\frac{2e^{j\omega}}{2-j\omega}\right]\right]_{\omega=0}$

$\frac{d}{d\omega}: \frac{j2e^{j\omega}}{2-j\omega} + \frac{2e^{j\omega}}{(2-j\omega)^2}\cdot j$

$E[X^2] = \frac{2}{2} + \frac{2}{4} = \boxed{3/2}$

$$E[X^2] = \frac{1}{j^2}\frac{d^2}{d\omega^2}\left[\phi_X(\omega)\right]_{\omega=0} = \frac{1}{j}\frac{d}{d\omega}\left[\frac{2e^{j\omega}}{2-j\omega} + \frac{2e^{j\omega}}{(2-j\omega)^2}\right]_{\omega=0}$$

$$= \frac{1}{j}\left[\frac{j2e^{j\omega}}{2-j\omega} + \frac{j2e^{j\omega}}{(2-j\omega)^2} + \frac{j2e^{j\omega}}{(2-j\omega)^2} + \frac{j2(2e^{j\omega})}{(2-j\omega)^3}\right]_{\omega=0}$$

$$= \frac{2}{2} + \frac{2}{4} + \frac{2}{4} + \frac{4}{8} = 3 \;/\; Var(X) = 3 - (3/2)^2 = \boxed{3/4}$$

Alternate:

$$\phi_X(\omega) = e^{j\omega-1}\left(1 + \frac{1}{1-j\omega}\right)$$

$$E[X] = \frac{1}{j}\frac{d}{d\omega}\left[\phi_X(\omega)\right]_{\omega=0}$$

$$= \frac{1}{j}\left[\frac{d}{d\omega}\left(e^{j\omega-1} + \frac{e^{j\omega-1}}{1-j\omega}\right)\right]_{\omega=0}$$

$$= \frac{1}{j}\left[je^{j\omega-1} + \frac{je^{j\omega-1}}{1-j\omega} + j\frac{e^{j\omega-1}}{(1-j\omega)^2}\right]_{\omega=0}$$

$$= \left[e^{-1} + \frac{e^{-1}}{1} + \frac{e^{-1}}{1}\right] = 3e^{-1}$$

$$E[X^2] = \frac{1}{j^2}\frac{d^2}{d\omega^2}\phi_X(\omega)\Big|_{\omega=0}$$

$$= \frac{1}{j}\left[\frac{d}{d\omega}e^{j\omega-1} + \frac{e^{j\omega-1}}{1-j\omega} + \frac{e^{j\omega-1}}{(1-j\omega)^2}\right]_{\omega=0}$$

$$= \frac{1}{j}\left[je^{j\omega-1} + \frac{je^{j\omega-1}}{1-j\omega} + \frac{je^{j\omega-1}}{(1-j\omega)^2} + \frac{je^{j\omega-1}}{(1-j\omega)^2} + \frac{j2e^{j\omega-1}}{(1-j\omega)^3}\right]_{\omega=0}$$

$$= e^{-1} + e^{-1} + e^{-1} + e^{-1} + 2e^{-1}$$

$$= 6e^{-1}$$

$$Var(X) = 6e^{-1} - (3e^{-1})^2 = \boxed{6e^{-1} - 9e^{-2}}$$

Question 5: [30%]

Consider the following game. You choose a number a first. Then the computer generates a number X that is an exponential random variable with $\lambda = 0.5$. If $X > a$, then you receive a reward of a dollars. If $X \le a$, then you receive nothing.

1. [8%] What is the expected reward of this game in terms of a? Hint: It might be easier to first consider a special case $a = 3$. In this case, your reward function is

$$f(x) = \begin{cases} 3 & \text{if } x > 3 \\ 0 & \text{if } x \le 3 \end{cases}. \tag{4}$$

   You are interested in the expected reward $E(f(X))$.

2. [3%] If you play this game, what value of a should you choose? (If you do not know the answer to the previous question, you can assume the expected reward is $4ae^{-2a}$.) Hint: Maximize your expected reward by considering the first order derivative. (not on exam)

3. [14%] Suppose an oracle tells you that the next time you play this game, X must be greater than 2. Given that $X > 2$, what is the conditional expected reward of this game in terms of a.

4. [5%] How to tell whether a R.V. is discrete, or continuous, or of mixed type form its cdf? Plot *any* one cdf $F_X(x)$ that corresponds to a random variable of mixed type.

1. R = reward

$$R = \begin{cases} a, & X > a \\ 0, & X \le a \end{cases}$$

$$E[R] = a\int_a^\infty \lambda e^{-\lambda x}\,dx = \boxed{ae^{-0.5a}}$$

$\cancel{2}$. $\frac{d}{da} ae^{-0.5a} = e^{-0.5a} + (-0.5a\, e^{-0.5a}) = 0$

(not on exam)

$$e^{-0.5a} = 0.5ae^{-0.5a}$$

$$0.5a = 1$$

$$\boxed{a = 2}$$

3) $E[R \mid X>2]$

$$= a\int_a^{\infty} f_X(x \mid X>2)\,dx$$

$$f_X(x \mid X>2) = \frac{d}{dx} F_X(x \mid X>2)$$

$$F_X(x \mid X>2) = \frac{P(X \le x \cap X>2)}{P(X>2)}$$

$$= \frac{P(2 < X \le x)}{e^{-2(0.5)}}$$

$$= \frac{e^{-1} - e^{-0.5x}}{e^{-1}} = 1 - e^{-(0.5x-1)}$$

$$f_X(x \mid X>2) = 0.5e^{1}e^{-0.5x}$$

$$\therefore E[R \mid X>2] = a\int_a^{\infty} 0.5e^{1}e^{-0.5x}\,dx \quad , a>2$$

$$= 0.5e^{1}a\left[-2e^{-0.5x}\right]_a^{\infty}$$

$$= 0.5e^{1}a\left(0 + 2e^{-0.5a}\right)$$

$$= \boxed{ae^{-(0.5a-1)}}$$

4) A cdf is discrete if it's cdf is a stair-step pattern, continuous if it is a smooth function with no discontinuities, and mixed if it contains both of these aspects