**ECE 302, Midterm #2**

6:30-7:30pm Thursday, March 5, 2009, EE 170,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains only work-out questions. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 12 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.
6. You can rip off the table in the back of the exam booklet.

Name:

Student ID:

E-mail:

Signature:

*Question 1:* [30%]

1. [6%] $X$ is an exponential random variable with $\lambda = 2$. Find $E((X - 0.5)^2)$. Hint: Find out $E(X)$ first.

2. [7%] $X$ is a binomial random variable with $n = 100$, $p = 0.1$. Find $P(X = 0|X < 2)$.

3. [8%] $X$ is a Poisson random variable with $\alpha = 1$. $f(x)$ is a function such that

$$f(x) = \begin{cases} 3 & \text{if } -0.1 < x \leq 1.5 \\ -2 & \text{if } 1.5 < x \leq 1.9 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   Find $E(f(X)^2)$.

4. [9%] $X$ is a geometric random variable with $p = 4/5$. Find $E(2^X - X)$.

*Question 2:* [18%] Let $X$ be a Bernoulli random variable with $p = 1/3$. Let $Y$ be a binomial random variable with $n = 2$ and $p = 1/4$. Suppose $X$ and $Y$ are independent. Answer the following questions:

1. [3%] What is the sample space when considering jointly $(X, Y)$? What is the corresponding weight assignment?

2. [10%] Let $Z = X^3 Y$. Find and plot the cdf $F_Z(z)$.

3. [5%] Find and plot the generalized pdf $f_Z(z)$. If you do not know the answer to the previous question, you can assume $F_Z(z)$ being as follows.

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 0.25\sin(z) & \text{if } 0 \leq z < \frac{\pi}{2} \\ \frac{z}{\pi} & \text{if } \frac{\pi}{2} \leq z < \pi \\ 1 & \text{if } \pi \leq z \end{cases} \tag{2}$$

*Question 3:* [5%] Consider a discrete random variable $W$ with cdf $F_W(w)$ being

$$F_W(w) = \begin{cases} 0 & \text{if } w < 0.5 \\ 0.01w^2 & \text{if } 0.5 \leq w < 2\pi \\ 1 - \sin(\frac{1}{w^2}) & \text{if } 2\pi \leq w \end{cases} \tag{3}$$

Find the probability $P(0.5 \leq W < 4 \text{ or } 5.5 < W)$. Hint: You do not need to expand the solution (for example if your solution contains $0.01 \times 0.6^2$), and you can just leave it as is.

*Question 4:* [22%] Consider a random variable $X$ such that $S = (1, \infty)$, i.e., $X$ can be any larger-than-one real numbers (say 1.1, $\pi$, etc.) but $X$ cannot be smaller than one. We also know that the probability density of $X$ is $f_X(x) = ce^{-2x}$ for $x > 1$ where $c$ is a constant.

1. [3%] Find the $c$ value.

2. [10%] Find the characteristic function $\Phi_X(\omega)$.

3. [9%] Use the moment theorem to find the mean ([6%]) and the variance ([3%]) of $X$. (If you do not know the answer of the previous question, you may assume $\Phi_X(\omega) = e^{j\omega-1}(1 + \frac{1}{1-j\omega})$.)

   Hint: Computer the variance takes a lot of time. Make sure you compute the mean first and then go to other questions. When you really have time, you can come back and solve the variance.

*Question 5:* [25%]

Consider the following game. You choose a number $a$ first. Then the computer generates a number $X$ that is an exponential random variable with $\lambda = 0.5$. If $X > a$, then you receive a reward of $a$ dollars. If $X \leq a$, then you receive nothing.

1. [7%] What is the expected reward of this game in terms of $a$? Hint: It might be easier to first consider a special case $a = 3$. In this case, your reward function is

$$f(x) = \begin{cases} 3 & \text{if } x > 3 \\ 0 & \text{if } x \leq 3 \end{cases}. \tag{4}$$

   You are interested in the expected reward $E(f(X))$.

2. [13%] Suppose an oracle tells you that the next time you play this game, $X$ must be greater than 2. Given that $X > 2$, what is the conditional expected reward of this game in terms of $a$.

3. [5%] How to tell whether a R.V. is discrete, or continuous, or of mixed type form its cdf? Plot *any* one cdf $F_X(x)$ that corresponds to a random variable of mixed type.

**ECE 302, Summary of Random Variables**

## Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\text{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\text{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\text{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega} - 1)}$, $G_X(z) = e^{\alpha(z-1)}$.

**Continuous Random Variables**

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.