ECE 302, Midterm #1

6:30-7:30pm Thursday, February 5, 2009, EE 170,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 13 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution

Student ID:

E-mail:

Signature:

Question 1: [15%]

Consider a sequence p_k such that for any integer k, we have

$$p_k = \frac{3^{-|k|}}{2}.$$

Let $F(m) = \sum_{k=-\infty}^{m} p_k$.

1. [8%] Find the expression of $F(m)$ for $m \leq 0$.
2. [7%] Find the value of $\sum_{k=-\infty}^{\infty} p_k e^{0.1k}$.

   Hint: The infinite sum of a geometric series is as follows.

$$\sum_{k=0}^{\infty} b_0 r^k = \frac{b_0}{1-r} \text{ if } |r| < 1. \tag{1}$$

1. $F(m)$ for $m \leq 0$:

$$F(m) = \sum_{k=-\infty}^{m} p_k = \sum_{k=-\infty}^{m} \frac{3^{-|k|}}{2}$$

Since $m \leq 0$, $k \leq 0$, $|k| = -k$.

$$\sum_{k=-\infty}^{m} \frac{3^{-|k|}}{2} = \frac{1}{2} \sum_{k=-\infty}^{m} 3^{k}$$

Let $l = -k$,

$$\frac{1}{2} \sum_{k=-\infty}^{m} 3^{k} \longrightarrow \frac{1}{2} \sum_{l=-m}^{\infty} 3^{-l} = \frac{1}{2} \sum_{l=-m}^{\infty} \left(\frac{1}{3}\right)^{l}$$

Now, let $j = l + m$

$$\frac{1}{2} \sum_{l=-m}^{\infty} \left(\frac{1}{3}\right)^{l} \longrightarrow \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^{j-m}$$

$$= \frac{1}{2} \left(\frac{1}{3}\right)^{-m} \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^{j}$$

$$= \frac{1}{2}(3^m) \cdot \frac{1}{1-\frac{1}{3}} = \frac{3^m}{2} \cdot \frac{3}{2}$$

$$= \boxed{3^{m+1}/4}$$

2. $$\sum_{k=-\infty}^{\infty} p_k e^{0.1k} = \sum_{k=-\infty}^{\infty} \frac{3^{-|k|}}{2} e^{0.1k}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} 3^{-k} e^{0.1k} + \sum_{k=-\infty}^{-1} 3^{k} e^{0.1k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k (e^{0.1})^k + \sum_{k=1}^{\infty} 3^{-k} e^{-0.1k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k (e^{0.1})^k + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k (e^{-0.1})^k - 1\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \left(\frac{1}{3}e^{0.1}\right)^k + \sum_{k=0}^{\infty} \left(\frac{1}{3}e^{-0.1}\right)^k - 1\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{3}e^{0.1}} + \frac{1}{1-\frac{1}{3}e^{-0.1}} - 1\right]$$

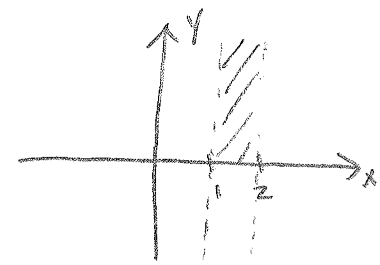
Question 2: [10%] Consider a 2-D function

$$f(x,y) = \begin{cases} xe^{-xy} & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Find out the value of $\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} yf(x,y)dydx$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dy\, dx$$

$f(x,y)$:



$$= \int_1^2 \int_0^{\infty} y x e^{-xy}\, dy\, dx = \int_1^2 x \left[\int_0^{\infty} y e^{-xy}\, dy\right] dx$$

$\int_0^{\infty} ye^{-xy}\, dy$: $\quad u = y \quad dv = e^{-xy}$

$\quad du = dy \quad v = -\frac{1}{x}e^{-xy}$

$$uv - \int_0^{\infty} v\, du = -y/x\, e^{-xy}\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{x} e^{-xy}\, dy$$

$$= (0-0) + \left[-\frac{1}{x^2}e^{-xy}\right]_0^{\infty} = \left(0 + \frac{1}{x^2}\right)$$

$$\therefore \int\int y f(x,y)\, dx\, dy = \int_1^2 x \left[\frac{1}{x^2}\right] dx = \ln x \Big|_1^2$$

$$= \boxed{\ln 2}$$

Question 3: [15%] Consider a 1-D function $f(x)$ such that

$$f(x) = \begin{cases} 0.25 & \text{if } -1 \le x < 2 \text{ or } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

1. [5%] Find out the value of $\int_{-\infty}^{\infty} x f(x) dx$.

2. [10%] Let $F(x) = \int_{-\infty}^{x} f(x) dx$. Find out the expressions of $F(x)$ for the following ranges: (1) when $2 \le x < 4$, and (2) when $4 \le x < 5$.

1. $$\int_{-\infty}^{\infty} x F(x)\, dx = \int_{-1}^{2} 0.25x\, dx + \int_{4}^{5} 0.25x\, dx$$

$$= \frac{0.25}{2} x^2 \Big|_{-1}^{2} + \frac{0.25}{2} x^2 \Big|_{4}^{5}$$

$$= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{25}{8} - \frac{16}{8}\right)$$

$$= \frac{12}{8} = \boxed{\frac{3}{2}}$$

2. (1) $$\underset{(2 \le x < 4)}{F(x)} = \int_{-1}^{2} 0.25\, dx = \boxed{3/4 \quad \text{for} \quad 2 \le x < 4}$$

(2) $$\underset{(4 \le x < 5)}{F(x)} = \int_{-1}^{2} 0.25\, dx + \int_{4}^{x} 0.25\, dx = \frac{3}{4} + \frac{1}{4}(x-4)$$

$$= \frac{3}{4} - 1 + \frac{1}{4}x$$

$$= \boxed{\frac{1}{4}x - \frac{1}{4}}$$

$$F(x) = \begin{cases} 3/4, & 2 \le x < 4 \\ \frac{1}{4}x - \frac{1}{4}, & 4 \le x < 5 \end{cases}$$

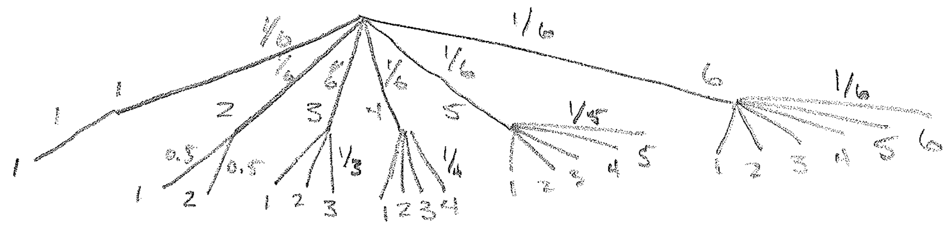
Question 4: [20%] Consider a sequential experiment such that Step 1: throw a fair 6-faced dice and denote the outcome as X. Step 2: Based on the value of X, a computer will choose a value Y from the integers between 1 and X with equal probability. (For example, if the output of $X = 3$, then the computer will choose randomly from the three integers $\{1, 2, 3\}$ with equal probability.)

1. [5%] What is the sample space when considering (X, Y) jointly?
2. [5%] What is the corresponding weight assignment? (Hint: The tree method may be more straightforward for this question.)
3. [5%] Find out the probability $P(X^2 + Y^2 < 10)$.
4. [5%] Find out the probability that $P(Y \geq 2 | X^2 + Y^2 < 10)$.

1. $S = \{(X,Y) : X \in \{1,2,\dots,6\};\ Y \in \{1,2,\dots,X\}\}$

$= \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

2.



X \ Y	1	2	3	4	5	6
1	1/6	—	—	—	—	—
2	1/12	1/12	—	—	—	—
3	1/18	1/18	1/18	—	—	—
4	1/24	1/24	1/24	1/24	—	—
5	1/30	1/30	1/30	1/30	1/30	—
6	1/36	1/36	1/36	1/36	1/36	1/36

3. $P(X^2+Y^2 < 10) = P(\{(1,1),(2,1),(2,2)\})$

$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \boxed{\frac{1}{3}}$$

4. $P(Y \geq 2 \mid X^2+Y^2 < 10) = \dfrac{P(\{Y \geq 2\} \cap \{X^2+Y^2 < 10\})}{P(\{X^2+Y^2<10\})}$

$$= \frac{P(\{(2,2)\})}{\frac{1}{3}} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{3}{12} = \boxed{\frac{1}{4}}$$

Question 5: [20%]

1. [3%] Write down the definition of "an event."

2. [3%] Write down the definition of "the sample space."

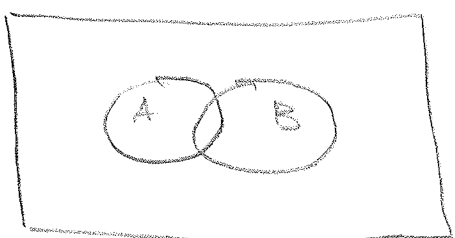
3. [14%] Consider two events A and B such that $P(A) = 1/2$, $P(A^c \cup B) = 3/4$, and $P(A \cap B) = 1/4$. Find out the probability $P(B^c)$. Hint: Use the Venn diagram to carefully identify the weight behind each region. (You should consider and identify the weights for the following four disjoint regions $A \cap B$, $A \cap B^c$, $A^c \cap B$, and $A^c \cap B^c$.)

$P(A \cup B) = 5/6$

1. A specified set of outcomes.

2. The set of all possible outcomes.

3.



$P(A) = 1/2$

$P(A^c \cup B) = 3/4$

$P(A \cup B) = 5/6$

$P(A \cap B) = 1/4$

$$P(B) = P(A \cup B) - P(A) + P(A \cap B)$$
$$= 5/6 - 1/2 + 1/4 = 7/12$$

$$P(B^c) = 1 - P(B) = \boxed{5/12}$$

Question 6: [20%] The computer *uniformly randomly* chooses a real number X from the interval $[1, 4]$.

1. [3%] What is the sample space?

2. [5%] What is the weight assignment? (Note that X is a continuous random variable. So you need to specify the corresponding pdf.)

3. [6%] What is the probability that the integer part of X is an odd integer. (For example, if $X = 3.1415$, then the integer part of X is 3, which is indeed an odd integer.)

4. [6%] Conditioning on that the integer part of X is odd, what is the probability that the integer part of $2X$ is also odd. (For example, if $X = 3.1415$, then $2X = 6.2830$ and the integer part of $2X$ is 6, which is not odd.)

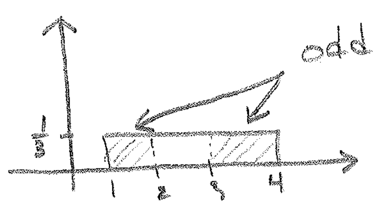
1. $S = \{x : x \in [1, 4]\}$

2. $f_X(x) = \dfrac{1}{b-a}$ for $a \le x \le b$ (Uniform r.v.)

In this case, $b = 4$, $a = 1$.

$f_X(x) = \dfrac{1}{3}$ for $1 \le x \le 4$

3.



$P(\text{integer}(x) \text{ is odd}) = \displaystyle\int_1^2 \tfrac{1}{3}\,dx + \int_3^4 \tfrac{1}{3}\,dx = \boxed{\tfrac{2}{3}}$

4. integer $(2X)$ is odd when $\begin{cases} 1.5 \le x < 2 \\ 2.5 \le x < 3 \\ 3.5 \le x < 4 \end{cases}$

$P(\text{integer}(2x) \text{ is odd} \mid \text{integer}(x) \text{ is odd}) =$

$$\frac{P(\{\text{integer}(2x) \text{ is odd}\} \cap \{\text{integer}(x) \text{ is odd}\})}{P(\{\text{integer}(x) \text{ is odd}\})}$$

$$P(\{\text{integer}(2x) \text{ odd}\} \cap \{\text{integer}(x) \text{ odd}\}) = \int_{1.5}^{2} \frac{1}{3}\,dx + \int_{3.5}^{4} \frac{1}{3}\,dx$$

$$= \frac{1}{3}$$

$$\therefore\ P(\text{integer}(2x) \text{ odd} \mid \text{integer}(x) \text{ odd}) = \frac{\frac{1}{3}}{\frac{2}{3}} = \boxed{\frac{1}{2}}$$