ECE 302, final exam of the session of Prof. Chih-Chun Wang

Wednesday 8–10am , May 6, 2009, Lily 1105,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. The total points of this exam are 200.
4. This exam contains both multiple-choice and work-out questions. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
5. There are 20 pages in the exam booklet. Use the back of each page for rough work. The last two pages are the Tables. You may rip the last two pages for easier reference. **Do not use your own copy of the Tables. Using your own copy of Tables will be considered as cheating.**
6. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

Signature:

Question 1: [30 pts] (out of 200 pts)

We know that the average number of customers per hour is 5. Let X denote the number of customers arrive in the morning from 9am-12pm.

1. [6 pts] What type of random variable is X? What is the value(s) of the parameter(s) of the distribution of X?

2. [10 pts] What is the probability that $P(\min(X^2, 7) < 5)$? (If you do not know the answer to the first sub question, you can assume X is geometric with $p = 1/16$.)

3. [8 pts] Suppose each customer spends \$100. What are the mean and the variance of the total number of money spent in the morning (between 9am-12pm)? (If you do not know the answer to the first sub question, you can assume X is geometric with $p = 1/16$.)

4. [6 pts] Use the Chebyshev inequality to upper bound $P(X \leq 5, \text{ or } 25 \leq X)$. (If you do not know the answer to the first sub question, you can assume X is geometric with $p = 1/16$.)

Question 2: [26 pts]

A salesperson traveled between cities A, B, and C for three consecutive nights, and each night he could only stay in one city. At day 1, the salesperson started from city A and stayed there for the first night. For the next two mornings, the salesperson randomly selected the next destination from the two other cities (excluding the city he stayed for the last night) with equaly probability. For example, if he stayed in city B for the second night, then he could spend the third night in either city A or city C with equal probability.

1. [10 pts] Construct the probability assignment for this random experiment.
2. [8 pts] Let X denote the number of nights that the salesperson stays in city A. What is the sample space and probability assignment for X.
3. [8 pts] Let Y denote the number of nights that the salesperson stays in one of cities B and C. Find $E(XY)$.

Question 3: [25 pts]

(X, Y) are uniformly chosen from the $(-1, 1) \times (-1, 1)$ square. (Namely, the outcome can be any (x, y) with $-1 < x < 1$ and $-1 < y < 1$.)

1. [5 pts] What is the joint pdf of (X, Y)?
2. [10 pts] What is the marginal pdf of X.
3. [10 pts] [Intermediate] Let $Z = X + Y$. What is the cdf of Z?

Question 4: [19 pts]

Let $g(t) = \cos(2\pi t)$. A random process $X(t)$ is described as $X(t) = Wg(t)$ where W is exponentially distributed with $\lambda = 2$.

1. [4 pts] Plot any two sample paths of $X(t)$ for $t = -1$ to 1.
2. [7 pts] Find out the value of $m_X(-0.5)$ where $m_X(t)$ is the mean function.
3. [8 pts] [Intermediate] Find out the value of $R_X(0.5, 1)$ where $R_X(t_1, t_2)$ is the auto correlation function.

Question 5: [32 pts] (out of 200 pts) For this question, there is no need to write down justifications. Choose from the following answers: Yes (the statement is always true), No (the statement is always false), or Not Necessary (sometimes it is true and sometimes it is false). It is recommended that you do not spend too much time on this question as we have 8 questions for this final.

1. [4 pts] Both X and Y are (marginally) Gaussian random variables. Are X and Y jointly Gaussian?

2. [4 pts] X and Y are jointly Gaussian. Is $W = 2X - 3Y$ Gaussian?

3. [4 pts] X and Y are jointly Gaussian and $W = X + Y$. Are X and W independent?

4. [4 pts] X and Y are jointly Gaussian with $m_X = 1$, $\sigma_X^2 = 1$, $m_Y = 1$, $\sigma_Y^2 = 1$, and their *covariance* being zero. Are X and Y independent?

5. [4 pts] X and Y are jointly Gaussian with $m_X = 1$, $\sigma_X^2 = 1$, $m_Y = 1$, $\sigma_Y^2 = 1$, and their *correlation* being zero. Are X and Y independent?

6. [4 pts] Continue from the previous question. Given $Y = 1$, is the conditional distribution of X Gaussian?

7. [4 pts] X and N are jointly Gaussian with $m_X = 0$, $\sigma_X^2 = 1$, $m_N = 0$, $\sigma_N^2 = 1$, and X and N are independent. $Y = X + N$ and $W = X - N$. Are Y and W independent?

8. [4 pts] X is Bernoulli with parameter $p = 0.5$. N is standard Gaussian. X and N are independent. Is $Y = (-1)^X N$ Gaussian?

Question 6: [30 pts] (out of 200 pts)

The first computer generates a random number X that is uniformly distributed on $(0, 1)$. Given $X = x$, the second computer generates a Y that is uniformly distributed on $[0, x]$.

1. [5 pts] What is the joint sample space of (X, Y)? What is the joint pdf $f_{X,Y}(x, y)$? Your answer should be of the following form:

$$f_{X,Y}(x,y) = \begin{cases} \dots & \text{if} \dots \\ \dots & \text{if} \dots. \\ \dots & \text{if} \dots \end{cases} \tag{1}$$

2. [10 pts] Given $Y = 0.3$, what is the maximum likelihood (ML) detector $\hat{X}_{\mathrm{ML}}(0.3)$?

3. [10 pts] [Intermediate] Given $Y = 0.5$, what is the maximum a posteriori probability detector $\hat{X}_{\mathrm{MAP}}(0.5)$?

4. [5 pts] [Intermediate] Given $Y = 0.5$, what is the linear minimum mean square error (MMSE) estimator $\hat{X}_{\mathrm{lin,MMSE}}(0.5)$?

Question 7: [24 pts] (out of 200 pts)

X_1, X_2, and X_3 are independent Bernoulli random variables with the same parameter $p = 1/3$. Let $Y_1 = X_1$, $Y_2 = 0.5(X_1 + X_2)$, and $Y_3 = 0.5(X_2 + X_3)$.

1. [9 pts] Find the means $E(Y_2)$ and $E(Y_3)$ and the covariance $\mathrm{Cov}(Y_2, Y_3)$.

2. [9 pts] Define $M = \frac{Y_1+Y_2+Y_3}{3}$. Find the expectation $E(M)$.

3. [6 pts] [Intermediate] Find the variance of M. Hint: X_1 to X_3 are independent but Y_1 to Y_3 are not.

Question 8: [14 pts] (out of 200 pts)

$X_1, X_2, \cdots, X_{100}$ are independently and identically distributed random variables. Each X_i is equally likely to take values in either -2 or 2. Let

$$S = X_1 + X_2 + \cdots + X_{100}. \tag{2}$$

1. [14 pts] Use the central limit theorem to approximate the probability $P(S > 50)$. You will need to express your answer by the $Q(x)$ function, where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$.

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\text{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\text{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\text{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.