SOLUTIONS

ECE 302, Midterm #3
7:00-8:00pm Thursday, March 29, FRNY G140,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains 4 questions. You have one hour to complete it. I will suggest not spending too much time on a single question, and work on those you know how to solve.
4. The sub-questions of a given question may not be listed from the easiest to the hardest. The best strategy is to take a look at all the sub-questions before starting to solve them.
5. There are totally 12 pages in the exam booklet. Use the back of each page for rough work.
6. **Neither calculators nor help sheets are allowed.**
7. Read through all of the problems first, and consult with the TA during the first 15 minutes. After that, no questions should be asked unless under special circumstances, which is at TA's discretion. You can also get a feel for how long each question might take after browsing through the entire question set. Good luck!

Name:

Student ID:

E-mail:

Signature:

Question 1: (30%) Consider a random variable X of mixed type with the following generalized pdf:

$$f_X(x) = 0.5\delta(x) + \begin{cases} 0.5e^{-x} & \text{if } 0 \le x \\ 0 & \text{if } x < 0 \end{cases}. \quad (1)$$

1. (5%) What is the meaning (or definition) of the "expectation"? What is the meaning (or definition) of the "variance"?
2. (5%) Find $E(X)$.
3. (5%) Find the characteristic function $\Phi_X(\omega)$.
4. (5%) Find the second moment of X. (Hint: it will be easier if solved by the moment theorem. However, you can still solve it by direct integration.)
5. (5%) Find the variance of X.
6. (5%) If $Y = 3X + 4$, find $E(Y)$, $\mathsf{Var}(Y)$, and $\Phi_Y(\omega)$.

1 $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 \cdot f_X(x)\, dx$$

2 $$E[X] = \int_{0^-}^{\infty} (0.5\,\delta(x) + 0.5\,e^{-x})\, x\, dx$$

$$= 0 \times 0.5 + 0.5 = 0.5 \;//$$

3 $$\Phi_X(\omega) = E(e^{j\omega X}) = \int_{0^-}^{\infty} e^{j\omega x} \cdot (0.5\,\delta(x) + 0.5 e^{-x})\, dx$$

$$= 0.5 + \frac{0.5}{(1 - j\omega)} \;//$$

4 $$E[X^2] = \frac{1}{j^2} \frac{d^2 \Phi_X(\omega)}{d\omega^2}\Bigg|_{\omega=0} = \frac{1}{j^2} \frac{2 \times 0.5 \times (j)^2}{(1 - j\omega)^3}\Bigg|_{\omega=0} = 1$$

5 $$\text{Var}(X) = E[X^2] - (E[X])^2 = 0.75 \;//$$

⑥ $Y = 3X + 4$

$E[Y] = 3E[X] + 4$

$= 5.5$ //

$Var(Y) = 3^2 \cdot Var(X) = 6.75$ //

$\Phi_y(\omega) = E[e^{j\omega(3x+4)}] = e^{j\omega 4} \cdot \Phi_x(3\omega)$

$\cancel{= 0.5\, e^{j\omega 4}\left[1 + \frac{1}{1-e^{-j\omega}}\right]}$

$= 0.5\, e^{j\omega 4}\left[1 + \frac{1}{1 - j\omega 3}\right]$

Question 2: (35%) Two numbers X and Y are independently selected from the interval [0,2] uniformly randomly.

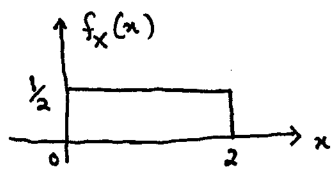
1. (5%) What are the marginal sample spaces for X and for Y respectively? Describe the marginal pdfs for X and for Y?
2. (5%) What is the joint sample space? What is the joint pdf?
3. (7%) Find the probability that the difference of X and Y is no larger than 1. Namely, find $P((X-Y) \leq 1)$. (Hint: In this specific setting, computing the "volume" instead of computing the double integration will save you a lot of time.)
4. (8%) Let $Z = \max(X, Y)$. Find the cdf of Z.
5. (5%) Find the pdf of Z. (If you do not know the answer to the previous question, you may assume $F_Z(z)$ as follows:

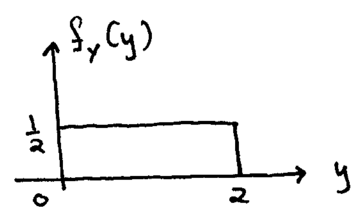
$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{8}z^3 & \text{if } 0 \leq z < 2 \\ 1 & \text{if } 2 \leq z \end{cases},$$

and continue solving the pdf of Z.)

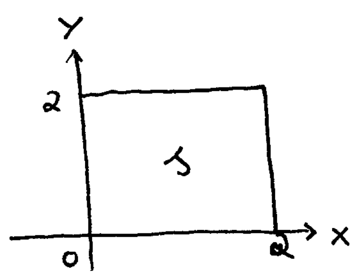
6. (5%) Find $E(|W|)$, where W is a Gaussian random variable with mean 0 and variance 1.

① $X \in [0,2]$ $\quad Y \in [0,2]$





② joint sample space :

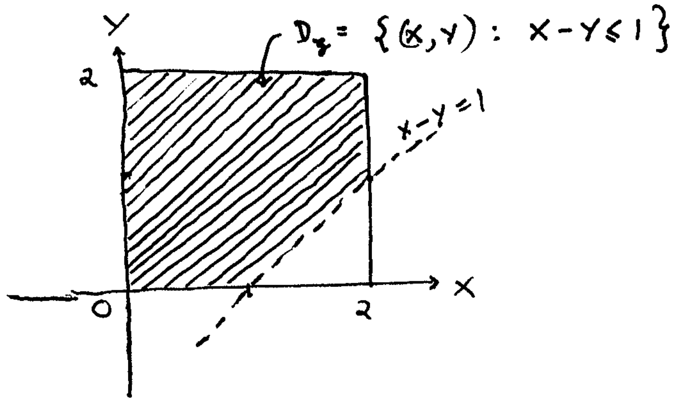


Since they are chosen independently

$F_{XY}(x,y) = F_X(x)\,F_Y(y)$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2}x & 0 < x \leq 2 \\ = 1 & x \geq 2 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{2}y & 0 < y \leq 2 \\ 1 & 2 < y \end{cases}$$



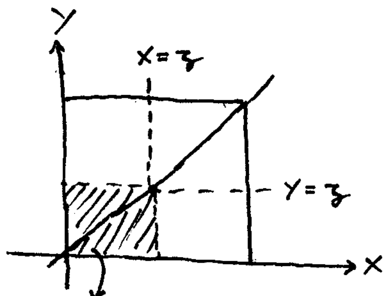
$$P(X - Y \le 1) = \iint_{D_z} f_{XY}(x,y)\,dx\,dy$$

$$= \cancel{4}\ 3 \times 1 \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times 1 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \quad //$$

(4) $Z = \max(X, Y)$

$$F_Z(z) = P(Z \le z)$$

$$= P(\max(X,Y) \le z)$$



$D_z = \{(x,y) : \max(x,y) \le z\}$

$$F_Z(z) = P((X,Y) \in D_z) = \frac{z^2}{4} \qquad 0 \le z \le 2$$

$$= 1 \qquad 2 < z$$

$$= 0 \qquad \text{o/w}$$

$$f_Z(z) = 0 \qquad z < 0$$

$$= \frac{z}{2} \qquad 0 \le z \le 2$$

$$= 0 \qquad 2 < z$$

(5) $E[\,|W|\,] = \quad 2 \cdot \int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x \cdot e^{-\frac{x^2}{2}}\, dx$

$$= \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} e^{-u}\, du = \frac{2}{\sqrt{2\pi}} \quad //$$

Question 3: (22%) Consider three magic dices X, Y, Z.

1. (2%) What is the joint sample space? (Hint: It can be considered as a big vector random variable.)

Suppose we know that the joint outcome (X, Y, Z) is uniformly distributed among the following 6 possible combinations $\{(1,2,3),(2,3,4),(3,4,5),(4,5,6),(5,6,1),(6,1,2)\}$ and has zero probabilistic weight for all other outcomes.

2. (5%) If a player is only able to observe the outcome of dice X, will the player think X is a fair dice? Why?

3. (5%) Are the outcomes of the two dices X and Y *independent*?

4. (5%) Consider the following game. The player pays \$40 for a chance to roll the three magic dices. Based on the outcome of (X, Y, Z), the player wins XYZ (the product of X, Y, and Z) dollars.

   What is the expected *net return* of the player?

5. (5%) What is the probability that the player is going to win some money?

① $S = \{ (x,y,z) \mid x,y,z \in \{1,2,3,\cdots,6\} \}$

② Yes. All the outcomes of X are equiprobable.

③ No. Infact

$$Y = \begin{cases} (X+1) & X \neq 6 \\ 1 & X = 6 \end{cases}$$

So there is a deterministic relation between X & Y.

④ The possible values of the return are:

6, 24, 60, 120, 30, 12 all with a probability of $\frac{1}{6}$

So the average return is 42.

So the expected net return is $42 - 40 = 2$ dollars.

(5) P(Player wins some money)

$$= P(\text{return is } 120 \text{ or } 60)$$

$$= \frac{1}{3} \quad //$$

Question 4: (18%) Suppose X is a random number that is exponentially distributed between $(0,\infty)$ with parameter $\lambda = 2$. Given X, Y is exponentially distributed between (X,∞), namely, the conditional pdf of Y given X is

$$f_{Y|X}(y|x) = \begin{cases} e^{-(y-x)} & \text{if } 0 \le x \le y < \infty \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

1. (6%) Find the joint pdf $f_{X,Y}(x,y)$.
2. (6%) Find the joint cdf $F_{X,Y}(x,y)$. There are three cases depending on the values of x and y. Case 1: If $x < 0$ or $y < 0$, then we know that $F_{X,Y}(x,y) = 0$. Case 2: If $0 \le x \le y < \infty$, then it is given that $F_{X,Y}(x,y) = (1-e^{-2x}) - 2e^{-y}(1-e^{-x})$. Find the expression of $F_{X,Y}(x,y)$ for the third case: $0 \le y \le x < \infty$. (Hint: It can be solved by direct integration or by observing the expressions for Cases 1 and 2.)
3. (6%) Find $P(X \le 2/3, 1/2 < Y \le 1)$.

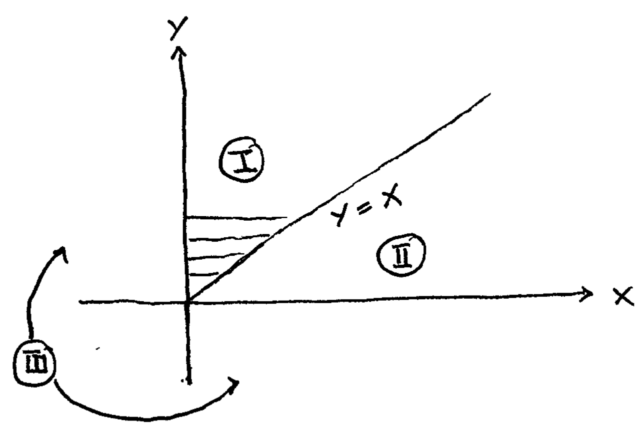
1

$$f_{X,Y}(x,y) = f_X(x)\cdot f_{Y|X}(y|x) \qquad (x,y)\in D$$

$$= 2e^{-2x}\cdot e^{-(y-x)} \qquad 0 \le x \le y < \infty$$

$$= 2e^{-(x+y)} \qquad 0 \le x \le y < \infty$$

2



if (x,y) is in region I $\quad (0 \le x \le y < \infty)$

$$F_{X,Y}(x,y) = (1-e^{-2x}) - 2e^{-y}(1-e^{-x})$$

if (x,y) is in region III $\quad F_{X,Y}(x,y) = 0$

if ~~Fx~~ (x,y) is in Region (II) $(0 \le y \le x < \infty)$

$$F_{XY}(x,y) = F_{XY}(y,y) \text{ from previous expression}$$

$$= 1 + e^{-2y} - 2e^{-y}$$

(3) $P\left(X \le \frac{2}{3} \,;\, \frac{1}{2} < Y \le 1\right)$

$$= F_{XY}\left(\tfrac{2}{3}, 1\right) - F_{XY}\left(\tfrac{2}{3}, \tfrac{1}{2}\right)$$

$$\left.\begin{aligned} &= \left(1 - e^{-\frac{4}{3}}\right) - 2e^{-1}\left(1 - e^{-2/3}\right) \\ &\qquad - \left(1 + e^{-1} - 2e^{-\frac{1}{2}}\right) \end{aligned}\right\} \text{from the previous part.}$$

$$= 2\left(e^{-\frac{5}{3}} + e^{-\frac{1}{2}}\right) - \left(e^{-4/3} + 3e^{-1}\right)$$