ECE 302, Midterm #2

7:00-8:00pm Thursday, Feb. 22, EE 170,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains 5 questions. You have one hour to complete it. I will suggest not spending too much time on a single question, and work on those you know how to solve.
4. The sub-questions of a given question are listed from the easiest to the hardest. The best strategy may be to finish only the sub-questions you know exactly how to solve.
5. There are a total of 11 pages in the exam booklet. Use the back of each page for rough work.
6. **Neither calculators nor help sheets are allowed.**
7. Read through all of the problems first, and consult with the TA during the first 15 minutes. After that, no questions should be asked unless under special circumstances, which is at TA's discretion. You can also get a feel for how long each question might take after browsing through the entire question set. Good luck!

Solution

Name:

Student ID:

E-mail:

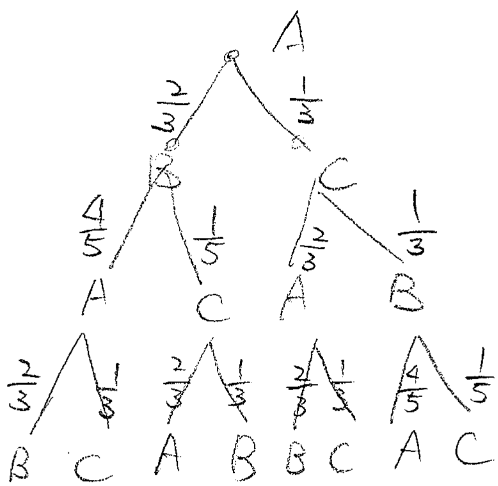
Signature:

Question 1: [20%] A salesperson travelled between cities A, B, and C for four consecutive nights, and each night he could only stay in one city. Suppose the populations of cities A, B, and C are 2, 1, and 0.5 millions respectively.

At day 1, the salesperson started from city A and stayed there for the first night. For the next three mornings, the salesperson randomly selected the next destination from the two cities excluding the city he stayed for the last night, and the probability he chose a certain city was proportional to the population of that city. For example, if he stayed in city B for the third night, then he could only spend the fourth night in either city A or city C. And the probabilities that he chose cities A and C are $P(A|B) = c \times 2$ and $P(C|B) = c \times 0.5$ for a common c.

1. [5%] What is the sample space?
2. [5%] Use the tree method to specify the weight assignment for the sample space.
3. [5%] What is the probability that the salesperson was able to visit all three cities?
4. [5%] What is the probability that the salesperson visited city B twice, given that the salesperson had visited all three cities?

1.



$S = \{ABAB, ABAC, ABCA, ABCB, ACAB, ACAC, ACBA, ACBC\}$

2.

$ABAB = \frac{2}{3} \times \frac{4}{5} \times \frac{2}{3} = \frac{16}{45}$

$ABAC = \frac{2}{3} \times \frac{4}{5} \times \frac{1}{3} = \frac{8}{45}$

$ABCA = \frac{2}{3} \times \frac{1}{5} \times \frac{2}{3} = \frac{4}{45}$

$ABCB = \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}$

$ACAB = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$

$ACAC = \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$

$ACBA = \frac{1}{3} \times \frac{1}{3} \times \frac{4}{5} = \frac{4}{45}$

$ACBC = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{5} = \frac{1}{45}$

3. $\frac{8}{45} + \frac{4}{45} + \frac{2}{45} + \frac{4}{27} + \frac{4}{45} + \frac{1}{45}$

$= 1 - \left(\frac{16}{45} + \frac{2}{27}\right)$

$= 1 - \left(\frac{48 + 10}{135}\right) = \frac{77}{135}$

4. $\frac{P(\text{All cities are visited and B is visited twice})}{P(\text{all cities are visited})}$

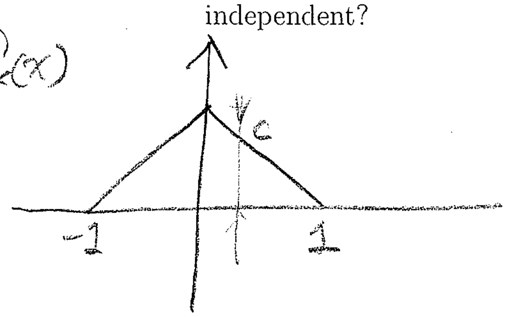
$= \frac{\frac{2}{45}}{\frac{77}{135}} = \frac{6}{77}$

Question 2: [20%] Consider a continuous random variable X with its pdf $f_X(x)$ as follows.

$$f_X(x) = \begin{cases} c(x+1) & \text{if } x \in [-1, 0] \\ c(1-x) & \text{if } x \in (0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

1. [5%] Find the value of c.
2. [5%] What is the definition of the cumulative distribution function (usually denoted by $F_X(x)$)?
3. [5%] Find the cdf $F_X(x)$.
4. [5%] Consider two events: $A = \{X \leq 0\}$ and $B = \left\{X^2 > \frac{1}{\sqrt{2}}\right\}$. Are A and B independent?

1. $f_X(x)$



$\Rightarrow \frac{1}{2} \times (1-(-1)) \times c = 1$

$\Rightarrow c = 1.$

2. $F_X(x) \triangleq P(X \leq x)$

3. $F_X(x) = \int_{-\infty}^{x} f_X(s)\,ds = \begin{cases} 0 & \text{if } x < -1 \\ \int_{-1}^{x} (s+1)\,ds = \frac{1}{2}(x+1)^2 & \text{if } -1 \leq x < 0 \\ \int_{-1}^{0} (1-s)\,ds + \int_{0}^{x} (1-s)\,ds = \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2}(1-x)^2\right) = 1 - \frac{1}{2}(1-x)^2 & \text{if } 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$

4. $P(A) = F_X(0) = \frac{1}{2}$

$$P(B) = P\left(X < -\frac{1}{\sqrt[4]{2}} \text{ or } X > \frac{1}{\sqrt[4]{2}}\right)$$

$$= P\left(X < -\frac{1}{\sqrt[4]{2}}\right) + P\left(X > \frac{1}{\sqrt[4]{2}}\right)$$

$$= \int_{-1}^{-\frac{1}{\sqrt[4]{2}}} (1+s)\,ds + \int_{\frac{1}{\sqrt[4]{2}}}^{1} (1-s)\,ds$$

$$= \frac{1}{2}\left(1-\frac{1}{\sqrt[4]{2}}\right)^2 + \frac{1}{2}\left(1-\frac{1}{\sqrt[4]{2}}\right)^2$$

$$= \left(1-\frac{1}{\sqrt[4]{2}}\right)^2$$

$$P(A \cap B) = P\left(X < -\frac{1}{\sqrt[4]{2}}\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{\sqrt[4]{2}}\right)^2 = P(A) \cdot P(B)$$

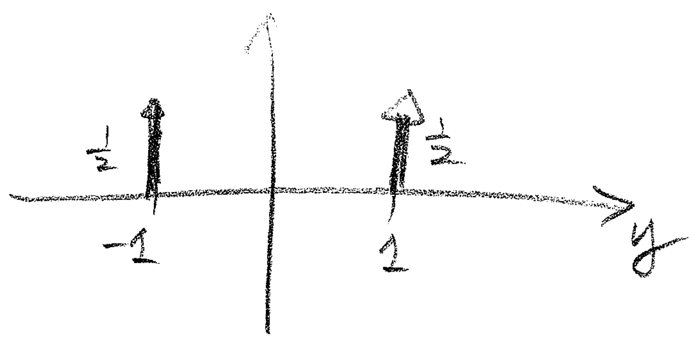
$\Rightarrow$ A & B are indep.

Question 3: [20%] Consider two independent random variables X and Y. X is a *continuous* uniform random variable distributed between $(-2, 2)$. Y is a *discrete* random variable with equal probability weight among $\{-1, 1\}$.

1. [5%] Find the pdf $f_Y(y)$. Express it as a single equation and plot $f_Y(y)$.
2. [5%] Suppose $Z = X+Y$. Find the cdf $F_Z(z)$. Hint: $P(Z \leq z) = P(Z \leq z \text{ and } X = -1) + P(Z \leq z \text{ and } X = 1)$.
3. [5%] Find the conditional probability $P(X = -1 | Z < 0.5)$.
4. [5%] Find the pdf $f_Z(z)$. If you do not know the answer to the above question, then you may assume the $F_Z(z)$ is as follows and solve the question accordingly. (Note this is an incorrect assumption.)

$$F_Z(z) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$$

1. $f_Y(y) = \frac{1}{2}\,\delta(y+1) + \frac{1}{2}\,\delta(y-1)$



2. $F_Z(z) = P(Z \leq z \text{ and } Y = -1)$
$+ P(Z \leq z \text{ and } Y = 1)$
$= \frac{1}{2} P(Z \leq z \mid Y = -1)$
$+ \frac{1}{2} P(Z \leq z \mid Y = 1)$
$= \frac{1}{2} P(X - 1 \leq z \mid Y = -1)$

$$+ \frac{1}{2} P(X+1 < z \mid Y=1)$$

$$= \begin{cases} \frac{1}{2}\times 0 + \frac{1}{2}\times 0 = 0 & \text{if} \quad z < -3 \\ \frac{1}{2}(z+1-(-2))\frac{1}{4} + \frac{1}{2}\times 0 & \text{if} \quad -3 \le z < -1 \\ = \frac{1}{8}(z+3) & \\ \frac{1}{2}(z+3)\times\frac{1}{4} + \frac{1}{2}(z+1)\times\frac{1}{4} & \text{if} \quad -1 \le z < 1 \\ = \frac{1}{4}z + \frac{1}{2} & \\ \frac{1}{2}\times 1 + \frac{1}{2}(z+1)\times\frac{1}{4} & 1 \le z < 3 \\ = \frac{1}{8}z + \frac{5}{8} & \\ \frac{1}{2}\times 1 + \frac{1}{2}\times 1 = 1 & 3 \le z \end{cases}$$

3. $P(Z < 0.5) = F_X(0.5) = \frac{5}{8}$

$$P(Z < 0.5 \text{ and } Y = -1) = P(X < 1.5 \text{ and } Y = -1)$$

$$= \frac{1}{4}\times(1.5-(-2))\times\frac{1}{2}$$

$$= \frac{7}{16}$$

$$\Rightarrow P(Y=-1 \mid Z < 0.5) = \frac{7/16}{5/8} = \frac{7}{10}$$

4. $$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \begin{cases} 0 & \text{if } z < 3 \\ \frac{1}{8} & \text{if } -3 \le z < -1 \\ \frac{1}{4} & \text{if } -1 \le z < 1 \\ \frac{1}{8} & \text{if } 1 \le z < 3 \\ 0 & \text{if } 3 \le z \end{cases}$$ #

Question 4: [20%] A factory is producing GPS chips. From the historic data, there are in average 20 defective chips per 24 hours being produced.

1. [10%] A consumer electronics company wants to place a big order of GPS chips and will pay this factory a visit. During the 3-hour visit, if there are more than 2 defective chips being produced, the company will cancel the order. What is the probability that the order will be cancel? You can assume the number of defective chips is Poisson distributed.

2. [10%] Find out the conditional probability that P(there are precisely 2 defective chips|the order is cancelled).

   But this time, we assume the number of defective chips is geometrically distributed with parameter $p = 0.9$.

1.

$$\alpha = \frac{20}{24} \times 3 = \frac{5}{2}$$

$$P(\text{cancel}) = P(X=3) + P(X=4) + \cdots$$

$$= 1 - \left(P(X=0) + P(X=1) + P(X=2)\right)$$

$$= 1 - \frac{(\frac{5}{2})^0}{0!} e^{-\frac{5}{2}} - \frac{(\frac{5}{2})^1}{1} e^{-\frac{5}{2}}$$

$$- \frac{(\frac{5}{2})^2}{2!} e^{-\frac{5}{2}}$$

$$= 1 - \left(\frac{7}{2} + \frac{25}{8}\right) e^{-\frac{5}{2}}$$

$$= 1 - \left(\frac{53}{8}\right) e^{-\frac{5}{2}}$$

2. Since when the order is cancelled
the # of defective chips ≥ 3
$\Rightarrow P(\text{\# of defective chip} = 2 \mid \text{cancel}) = 0$

Question 5: [25%] Consider X being the snow precipitation of February (with unit: "inch"). From the historical data (or we may assume that), X is a Gaussian distribution with $m = 20$, $\sigma = 10$. Your answers have to be computed/converted to numbers or rational numbers "a/b" instead of only writing down the integrals. You may want to use the included table of $\Phi(x)$, the cumulative distribution of a standard Gaussian distribution.

1. [5%] What is the definition of the standard Gaussian distribution?
2. [5%] Find the probability that $P(X \leq 20)$.
3. [5%] Find the probability that $P(20 \leq X < 40)$.
4. [5%] Find the probability that $P(5 \leq X < 10)$.
5. [5%] Conditioning that there have already been 25 inches of snow between 2/1–2/21, what is the probability that there will be an additional 10 inches of snow in the next week (between 2/22–2/28).

$\Phi(0.5) = 0.691$, $\Phi(1) = 0.841$, $\Phi(1.5) = 0.9332$, $\Phi(2) = 0.9662$, $\Phi(2.5) = 0.99379$, $\Phi(3) = 0.99865$.

1. A standard Gaussian is a Gaussian with $m = 0$ $\sigma = 1$.

2. X can be written as

$X = 20 + 10Z$ where Z is a standard Gaussian

$P(X \leq 20) = P(Z \leq 0) = \frac{1}{2}$

3. $P(20 \leq X < 40) = P(0 \leq Z < 2) = \Phi(2) - \Phi(0)$

$= 0.4662$

4. $P(5 \leq X < 10) = P(-1.5 \leq Z < 1)$

$= P(1 < Z \leq 1.5)$ (by symmetry)

$= \Phi(1.5) - \Phi(1) = 0.9332 - 0.841$

$= 0.0922$

5. $P(X > 35 \mid X > 25)$

$$= \frac{P(X > 35)}{P(X > 25)} = \frac{P(Z > 1.5)}{P(Z > 0.5)}$$

$$= \frac{1 - P(Z \leq 1.5)}{1 - P(Z \leq 0.5)}$$

$$= \frac{1 - \Phi(1.5)}{1 - \Phi(0.5)}$$

$$= \frac{1 - 0.9332}{1 - 0.691}$$

$$= \frac{0.0668}{0.309} \quad \#$$