ECE 302, Midterm #1
7:00-8:00pm Thursday, Jan. 25, EE 170,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains 7 questions. You have one hour to complete it. I will suggest not spending too much time on a single question, and work on those you know how to solve.
4. The sub-questions of a given question are listed from the easiest to the hardest. The best strategy may be to finish only the sub-questions you know exactly how to solve.
5. There are a total of 15 pages in the exam booklet. Use the back of each page for rough work.
6. **Neither calculators nor help sheets are allowed.**
7. Read through all of the problems first, and consult with the TA during the first 15 minutes. After that, no questions should be asked unless under special circumstances, which is at TA's discretion. You can also get a feel for how long each question might take after browsing through the entire question set. Good luck!

Solution

Name:

Student ID:

E-mail:

Signature:

Question 1: [7%] Suppose a_k is a series such that

$$a_k = \begin{cases} \frac{2}{3}\left(\frac{1}{3}\right)^k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

1. [7%] Find out the values of

$$\sum_{k=20}^{\infty} a_k$$

$$\sum_{k=-\infty}^{19} a_k.$$

Hint: The infinite sum of a geometric series is as follows.

$$\sum_{k=0}^{\infty} b_0 r^k = \frac{b_0}{1-r} \tag{1}$$

1. $\displaystyle\sum_{k=20}^{\infty} a_k = \sum_{k=20}^{\infty} \left(\frac{1}{3}\right)^k \times \left(\frac{2}{3}\right)$

$\displaystyle = \frac{\left(\frac{1}{3}\right)^{20} \times \left(\frac{2}{3}\right)}{1-\frac{1}{3}} = \left(\frac{1}{3}\right)^{20}$

2. $\displaystyle\sum_{k=-\infty}^{19} a_k = \sum_{k=-\infty}^{\infty} a_k - \sum_{k=20}^{\infty} a_k$

$\displaystyle = \frac{2/3}{1-\frac{1}{3}} - \left(\frac{1}{3}\right)^{20}$

$\displaystyle = 1 - \left(\frac{1}{3}\right)^{20}$

Question 2: [15%] Define a 1-D function $f_X(x)$ as follows.

$$f_X(x) = \begin{cases} \frac{1}{3} & \text{if } x \in [0,1] \\ \frac{2}{3} & \text{if } x \in (1,2] \\ 0 & \text{otherwise} \end{cases}$$

Another function $F(x)$ can be defined based on the integral of $f_X(x)$ as follows:

$$F(x) = \int_{s=-\infty}^{x} f_X(s)ds.$$

Hint: Solve the first two sub-questions first and go back to the third sub-question if you have time.

1. [5%] Find the value of $F(-1)$. (Hint: Do not be scared by this expression. This question is no different than asking you to compute the value of $\int_{-\infty}^{-1} f_X(s)ds$.)
2. [5%] Find the value of $F(0.5)$
3. [5%] Find the value of $F(x)$ assuming $x \in (1,2]$.

1. $F(-1) = \int_{-\infty}^{-1} f_X(s)\,ds = \int_{-\infty}^{-1} 0\,ds = 0$

2. $F(0.5) = \int_{-\infty}^{0.5} f_X(s)ds = \int_{0}^{0.5} \frac{1}{3}\,ds$

$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

3. $F(x) = \int_{-\infty}^{x} f_X(s)ds$

$= \int_0^1 \frac{1}{3} ds + \int_1^x \frac{2}{3} ds$

$= \frac{1}{3} + \frac{2}{3}(x-1)$

$= \frac{2}{3}x - \frac{1}{3}$

Question 3: [15%] Define a 1-D function $g_X(x)$ as follows.

$$g_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

1. [7.5%] Compute $\int_{x=-\infty}^{\infty} x g_X(x) dx$.
2. [7.5%] Compute the bilateral Laplace transform of $g_X(x)$.

   Hint 1: The bilateral Laplace transform of any function $f(x)$ is defined as

   $$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

   Hint 2: You can safely assume $|s| < 1$ during your computation.

1. $\int_{x=-\infty}^{\infty} x g_X(x) dx$

$$= \int_0^{\infty} x e^{-x} dx$$

$$= -\int_0^{\infty} x \, de^{-x} = -\left[xe^{-x} \Big|_0^{\infty} - \int_0^{\infty} e^{-x} dx \right]$$

$$= -[0 - 1] = 1$$

2. $L_g(s) = \int_{-\infty}^{\infty} e^{-sx} g_X(x) dx$

$$= \int_0^{\infty} e^{-sx} e^{-x} dx$$

$$= \frac{1}{-(s+1)} e^{-(s+1)x} \Big|_0^{\infty}$$

$$= \frac{1}{s+1}$$

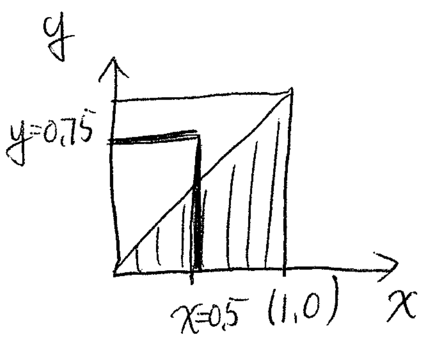
Hint: Solve Questions 5–7 first and come back to this question if you have time.

Question 4: [10%] Define a 2-D function $f(x,y)$ as follows.

$$f(x,y) = \begin{cases} 1/x & \text{if } x \in (0,1] \text{ and } y \in [0,x] \\ 0 & \text{otherwise} \end{cases}$$

1. [10%] Compute the value of the following 2-dimensional integral.

$$\int_{y=-\infty}^{0.75} \int_{x=-\infty}^{0.5} f(x,y)dxdy.$$



$$\int_{y=-\infty}^{0.75} \int_{x=-\infty}^{0.5} f(x,y)dxdy$$

$$= \int_{x=-\infty}^{0.5} \int_{y=-\infty}^{0.75} f(x,y)\,dy\,dx$$

$$= \int_{x=0}^{0.5} \underline{\int_{y=0}^{x} \frac{1}{x}\,dy}\,dx$$

$$= \int_{x=0}^{0.5} 1 \quad dx$$

$$= 0.5 \ \#$$

Question 5: [21%] Throw two *unfair* 6-faced dices and let X and Y denote the outcomes of each dice. We know that there is some invisible magnetic force between these two dices so that X will never be the same as Y. (Namely, the outcomes of these two dices will never be identical.) All other outcomes occur equally likely.

1. [3%] What is the definition of "sample space"?
2. [3%] What is the sample space in this experiment?
3. [4%] What is the probability weight you would like to assign to each outcome of the sample space?
4. [4%] What is the probability that $X^2 + Y$ is no larger than 10?
5. [4%] What is the $P(Y \le 2 | X^2 + Y \le 10)$?
6. [3%] What is the $P(X = 4 | X^2 + Y \le 10)$?

2. $S = \{(1,1), (1,2), \cdots (1,6)$

$(2,1), \cdots (2,6)$

$\vdots$

$(6,1), \cdots, (6,6)\}.$

1. The sample space is the collection of all possible outcomes.

3.

Y \ X	1	2	3	4	5	6
1	0	W	W	W	W	W
2	W	0	W	W	W	W
3	W	W	0	W	W	W
4	W	W	W	0	W	W
5	W	W	W	W	0	W
6	W	W	W	W	W	0

$30W = 1$

$W = \frac{1}{30}$

4. The event contains

$(1,1), \cdots, (1,6)$

$(2,1), \cdots, (2,6)$

$(3,1)$

$\Rightarrow P(X^2 + Y \le 10)$

$= 0 \times 2 + \frac{1}{30} \times 11 = \frac{11}{30}$

5. $\cancel{\{Y \le 2 \mid X^2 + Y \le 10}$

$\{Y \le 2\} \cap \{X^2 + Y \le 10\}$

$= \{(1,1)\ (1,2)\ (2,1)\ (2,2),\ (3,1)\}$

$P(Y \le 2 \text{ and } X^2 + Y \le 10)$

$= 0 \times 2 + 3 \times \frac{1}{30}$

$\Rightarrow P(Y \le 2 \mid X^2 + Y \le 10) = \frac{3/30}{11/30} = \frac{3}{11}$

6. Since $\{X = 4\} \cap \{X^2 + Y \le 10\} = \phi$ empty set

$\Rightarrow P(X = 4 \mid X^2 + Y \le 10) = 0$ #

Question 6: [8%] Consider an unfair 3-faced dice and each face has 4, 5, or 6 dots respectively. Throw this dice once and let X denote the number of dots that is facing up. Let A denote the event $X \leq 5$ and B denote the event $X \geq 5$. Suppose we also know that

$$P(A) = \frac{5}{6} \tag{3}$$

$$P(B) = \frac{1}{2}. \tag{4}$$

Find out the probability $P(X$ is not a prime number$)$.

$$P_4 \triangleq P(X=4)$$

$$P_5 \triangleq P(X=5)$$

$$P_6 \triangleq P(X=6)$$

$$\Rightarrow \begin{cases} P_4 + P_5 = \frac{5}{6} \\ P_5 + P_6 = \frac{1}{2} \\ P_4 + P_5 + P_6 = 1 \end{cases}$$

$$\Rightarrow \quad P_6 = \frac{1}{6} \quad P_5 = \frac{1}{3}$$

$$P_4 = \frac{1}{2}$$

$P(X$ is not a prime$)$

$= P\{X = 4 \text{ or } 6\}$

$= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

Question 7: [19%] A real number X is randomly drawn from $[0, 1]$. Answer the following question.

1. [2%] What is the sample space in this experiment?
2. [3%] How to specify a weight assignment for a discrete sample space? How to specify a weight assignment for a continuous sample space?
3. [3%] What is the common equation that the total sum of any weight assignment should satisfy?
4. [3%] The probability weight assignment in this question is specified by

$$f_X(x) = \begin{cases} c \cdot x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases},$$

for some unknown coefficient c. What is the value of c should be? Hint: Use the answer of the previous question.

5. [4%] What is the probability that $P(X < 0.5)$?
6. [4%] What is the conditional probability that $P(X > 0.25 | X < 0.5)$?

1. $S = [0, 1]$

2-1 Specify the weight for each outcome. such that the total sum $= 1$

2-2 Specify a curve such that the weight is the area underneath the curve. The total area should be 1.

3. The total sum of the weights $= 1$.

4. $\int_0^1 cx\,dx = \frac{1}{2}cx^2 \Big|_0^1 = \frac{1}{2}c = 1$

$\Rightarrow c = 2$.

5. $P(X < 0.5) = \int_0^{0.5} 2x\,dx = x^2 \Big|_0^{0.5} = \frac{1}{4}$

$P(X > 0.25 \text{ and } X < 0.5)$

$$= \int_{0.25}^{0.5} 2x\,dx = x^2 \Big|_{0.25}^{0.5}$$

$$= \frac{3}{16}$$

$\Rightarrow P(X > 0.25 \mid X < 0.5)$

$$= \frac{\frac{3}{16}}{\frac{1}{4}} = \frac{3}{4} \ \#$$