ECE 302 Division 2 (Instructor: Prof. Chih-Chun Wang) — Final Exam

1:00-3:00pm Wednesday, May 2, PHYS 114,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains 5 questions with a total of 150 points. You have two hours to complete it. I will suggest not spending too much time on a single question, and work on those you know how to solve.
4. The sub-questions of a given question ARE listed from the EASIEST to the HARDEST. The best strategy is to work on the sub-questions you know the way to solve it, and then change to the next question. Come back later if you have time.
5. There are totally 16 pages in the exam booklet. Use the back of each page for rough work.
6. **Neither calculators nor help sheets are allowed.**
7. Read through all of the problems first, and consult with the TA during the first 15 minutes. After that, no questions should be asked unless under special circumstances, which is at TA's discretion. You can also get a feel for how long each question might take after browsing through the entire question set. Good luck!

Name:

Student ID:

E-mail:

Signature:

Question 1: A bit X is chosen uniformly randomly from $\{0,1\}$. A binary non-symmetric channel is considered with the conditional distribution as follows:

$$
\begin{aligned}
P(Y=0|X=0) &= 1/4 \\
P(Y=1|X=0) &= 3/4 \\
P(Y=0|X=1) &= 1/3 \\
P(Y=1|X=1) &= 2/3.
\end{aligned}
$$

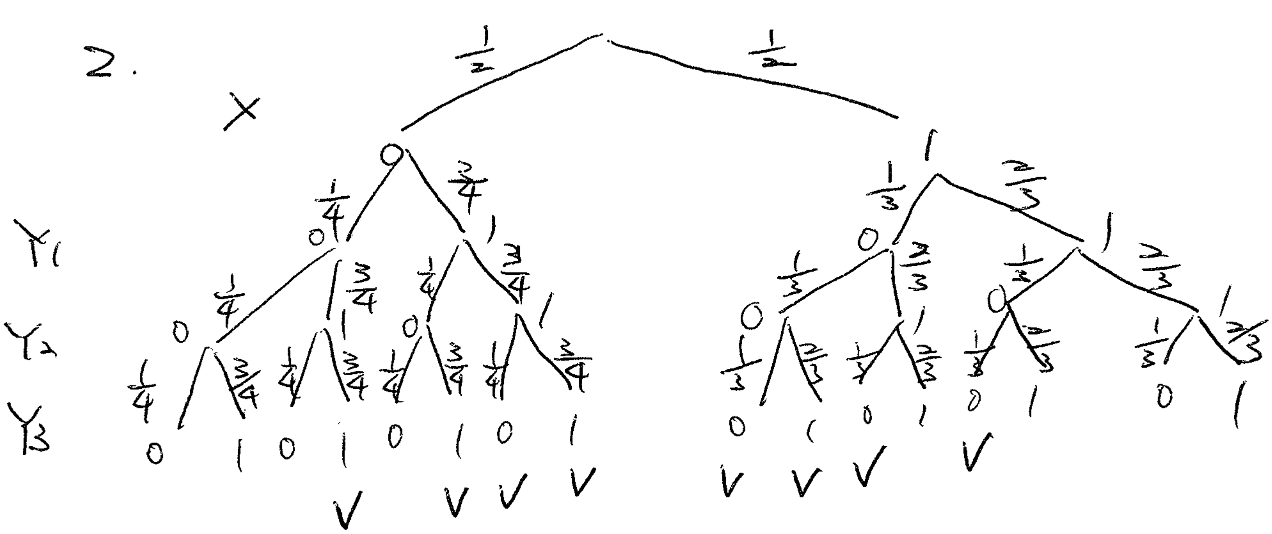
Suppose X is repeated three times, and sent through the BSC. Namely, we transmit (X,X,X) and receive (Y_1,Y_2,Y_3). We further assume that each BSC usage is independent.

1. (5%) What is the sample space of this experiment.
2. (10%) What is the weight assignment for this experiment. Hint: use the tree method.
3. (5%) The receiver uses the majority vote to determine $\hat{Y}$, the majority of (Y_1,Y_2,Y_3). For example, if $(Y_1,Y_2,Y_3) = (0,1,0)$, the majority vote $\hat{Y} = 0$. What is the probability that $P(X \neq \hat{Y})$?
4. (5%) What is the conditional probability of $X = 0$, given the received values are $Y_1, Y_2, Y_3 = 001$.
5. (5%) Is the majority vote a good decision rule for determining X from the observation (Y_1,Y_2,Y_3)? Why or why not?

1.

X \ Y	000	001	010	011	100	101	110	111
0								
1								

2.



3. $\hat{Y} \neq X$: V

$P(\hat{Y} \neq X) = \frac{1}{2}\left(\frac{1}{4}\right)\times\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)\times\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$

$$+\frac{1}{2}\left(\frac{3}{4}\right)^3+\frac{1}{2}\left(\frac{1}{3}\right)^3+\left(\frac{1}{2}\right)\times\left(\frac{1}{3}\right)^2\times\left(\frac{2}{3}\right)$$

$$+\frac{1}{2}\times\left(\frac{1}{3}\right)^2\times\left(\frac{2}{3}\right)+\frac{1}{2}\times\left(\frac{1}{3}\right)^2\times\left(\frac{2}{3}\right)$$

$$=\frac{9\times3+27}{128}+\frac{1+2\times3}{54}$$

$$=\frac{54}{128}+\frac{7}{54}=\frac{953}{1728}$$

4. $$\frac{P(X=0,\ Y_1Y_2Y_3=001)}{P(Y_1Y_2Y_3=001)}$$

$$=\frac{\frac{1}{2}\times\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\times\left(\frac{3}{4}\right)}{\frac{1}{2}\times\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\times\left(\frac{3}{4}\right)+\frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}$$

$$=\frac{\frac{3}{64}}{\frac{3}{64}+\frac{2}{27}}=\frac{81}{209}<50\%.$$

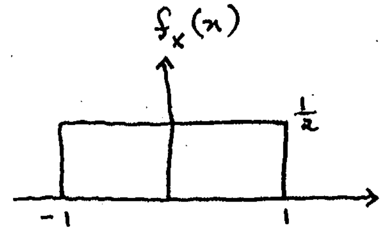
5. No. $\because$ The Conditional prob of X given Y_1, Y_2, Y_3 suggests that a reasonable decision rule should output $\hat{X}=1$. but the majority vote gives us $\hat{Y}=0$.

Question 2: (30%) Suppose a continuous random variable X is uniformly distributed between $(-1, 1)$. Suppose $Y = X^4$.

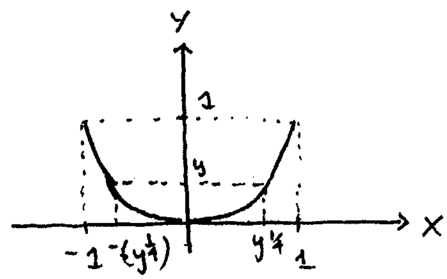
1. (5%) Find the mean, the variance of Y.
2. (5%) What is the definition of the cumulative distribution function (cdf) of Y?
3. (10%) Find the cdf and pdf of Y.
4. (10%) Find $E(Y/X)$ and $E(\cos(Y)|X|^3)$

Q2

Soln : $X \sim U(-1,1)$



$Y = X^4$



a) $E[Y] = E[X^4] = \int_{-1}^{1} \frac{1}{2} x^4 \, dx = \frac{1}{5}$ //

$E[Y^2] = E[X^8] = \int_{-1}^{1} \frac{1}{2} x^8 \, dx = \frac{1}{9}$ //

$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{16}{225}$ //

b) $F_Y(y) = P(Y \le y)$

c) from the figure

$F_Y(y) = P(Y \le y) = P(-\sqrt[4]{y} \le X \le \sqrt[4]{y})$ if $y \geq 0$

$= 0$ if $y < 0$

$F_Y(y) = y^{1/4}$ $\quad 0 \le y \le 1$

$= 1$ $\quad y > 1$

$= 0$ $\quad y < 0$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{4}\, y^{-3/4} \qquad 0 \leq y \leq 1$$

$$= 0 \qquad \text{otherwise} .$$

d) $$E\left[\frac{Y}{X}\right] = E\left[X^3\right] = \frac{1}{2}\int_{-1}^{1} x^3 \cdot dx = 0$$

$$E\left[\operatorname{Cos}(Y)\,|X|^3\right] = \int_{-1}^{0} -\frac{x^3 \cdot \operatorname{Cos}(x^4)}{2}\, dx + \int_{0}^{1} \frac{1}{2} \operatorname{Cos}(x^4)\, x^3 dx$$

$$= 2\int_{0}^{1} \frac{1}{8} \operatorname{Cos}(t) \cdot dt = \frac{1}{4}\left(\operatorname{Sin}(1)\right)$$

$$= \frac{1}{4}\left(\operatorname{Sin}(1)\right) \quad //$$

Question 3: (30%) Consider two Gaussian random variables X and Y with means and variances $(m_X = 2, \sigma_X^2 = 3)$ and $(m_Y = 2, \sigma_Y^2 = 4)$ respectively. We further assume that X and Y are independent.

1. (7%) Are X and Y orthogonal? Are X and Y correlated? Find out the correlation coefficient between X and Y.
2. (8%) Let $W = 3X - Y$. Find out the pdf $f_W(w)$, the mean m_W, and the variance σ_W^2 of W.
3. (5%) Find out the linear minimum mean square error estimator of X given W.
4. (5%) Find out the minimum mean square error estimator of W given $X = 1$.
5. (5%) Find out the minimum mean square error estimator of X given $W = 1$. Hint 1: This is a difficult question. Come back later if you have time. Hint 2: First solve the following sub-questions in sequence. Find $f_{W|X}$, $f_{W,X}$ and $f_{X|W}$.

1. $R(X,Y) = E(XY) = E(X)E(Y) = 2 \times 2 \neq 0$

$\Rightarrow$ not orthogonal

$C(X,Y) = E((X-m_X)(Y-m_Y)) = E(X-m_X) \cdot E(Y-m_Y)$

$= 0 \times 0 = 0$

$\Rightarrow$ uncorrelated.

$$\rho_{XY} = \frac{C(X,Y)}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0.$$

2. $m_W = 3m_X - m_Y = 3 \times 2 - 2 = 4.$

$\sigma_W^2 = 9\sigma_X^2 + \sigma_Y^2 = 27 + 4 = 31$

W is Gaussian

$$\Rightarrow f_W(w) = \frac{1}{\sqrt{2\pi \times 31}} e^{-\frac{(w-4)^2}{2 \times 31}}$$

3. $C(X,W) = E((X-m_x)(W-m_w))$

$= E(XW) - m_x m_w$

$= E(X(3X-Y)) - m_x m_w$

$= 3E(X^2) - E(X)E(Y) - m_x m_w$

$= 3\times(3+(2^2)) - 2\times 2 - 2\times 4$

$= 9$

$\rho_{XW} = \dfrac{C(X,W)}{\sigma_x \sigma_w} = \dfrac{9}{\sqrt{3\times 31}} = \dfrac{9}{\sqrt{93}}$

$\hat{X} = a^* W + b^*$

where $a^* = \rho_{XW} \dfrac{\sigma_x}{\sigma_w} = \dfrac{9}{\sqrt{93}} \times \dfrac{\sqrt{3}}{\sqrt{31}} = \dfrac{9}{31}$

$b^* = \rho_{XW} \dfrac{\sigma_x}{\sigma_w} \times (-m_w) + m_x$

$= -\dfrac{9}{31} \times 4 + 2 = \dfrac{26}{31}$

4. $\hat{W} = E(W \mid X=1)$

$= E(3\times 1 - Y \mid X=1)$

$= E(3-Y) = 3-2 = 1$

$$f_{W|X}(w|x) = \frac{1}{\sqrt{2\pi \times 4}} \times e^{-\frac{(w-(3x-2))^2}{2\times 4}}$$

$$f_{W,X}(w,x) = f_{W|X}(w|x) \cdot f_X(x)$$

$$= \frac{1}{\sqrt{2\pi \times 4}} \times e^{\frac{-(w-(3x-2))^2}{2\times 4}} \times \frac{1}{\sqrt{2\pi\times 3}} \times e^{-\frac{(x-2)^2}{2\times 3}}$$

$$f_{X|W}(x|w) = \frac{f_{W,X}(w,x)}{f_W(w)}$$

$$= \frac{1}{\sqrt{2\pi \frac{3\times 4}{31}}} e^{-\frac{\left(x - \frac{26+9w}{31}\right)^2}{2\times \frac{3\times 4}{31}}}$$

$$\Rightarrow \quad \tilde{X} = E(X|W=1) = \frac{35}{31} \quad \#$$

Question 4: (30%) Let $X_1, \cdots, X_n, \cdots$ be i.i.d. Poisson random variables with parameter $\alpha = 2$.

1. (5%) What does the acronym "i.i.d." stand for?

2. (8%) What is the definition of the characteristic function of X_1? Find out the characteristic function $\Phi_{X_1}(\omega)$.

3. (3%) Let $S_n = \sum_{i=1}^{n} X_i$. What is the characteristic function $\Phi_{S_n}(\omega)$? What type of random variable is S_n.

4. (6%) What are the mean and variance of S_n?

5. (8%) Use the central limit theorem to approximate the probability $P(370 < S_{200} < 460)$. You may need to use the following table of the cdf $\Phi(x)$ of a standard Gaussian random variable: $\Phi(0.5) = 0.691$, $\Phi(1) = 0.841$, $\Phi(1.5) = 0.9332$, $\Phi(2) = 0.9772$, $\Phi(2.5) = 0.99379$, $\Phi(3) = 0.99865$.

Q4

Solns $X_1, X_2, \cdots, X_n \cdots$ are i.i.d poisson RVs with parameter α.

a) i.i.d - independent & identically distributed.

b) $$\Phi_{X_1}(\omega) = E\left[e^{j\omega X_1}\right] = \sum_{k=0}^{\infty} \frac{\alpha^k e^{-\alpha}}{k!} \cdot e^{j\omega k}$$

$$= \sum_{k=0}^{\infty} \cancel{e^{-(\alpha - j\omega)}} \frac{(\alpha \cdot e^{j\omega})^k e^{-\alpha}}{k!}$$

$$= \left[\sum_{k=0}^{\infty} \frac{(\alpha \cdot e^{j\omega})}{k!} \cdot e^{-(\alpha \cdot e^{j\omega})}\right] e^{\alpha e^{j\omega}} \cdot e^{-\alpha}$$

$$\Phi_{X_1}(\omega) = e^{-\alpha(1 - e^{j\omega})}$$

c) $$S_n = \sum_{i=1}^{n} X_i$$

$$\Phi_{S_n}(\omega) = E\left(e^{j\omega S_n}\right) = E\left[\left(e^{j\omega\left(\sum_{i=1}^{n} X_i\right)}\right)\right]$$

$$= E\left[e^{j\omega(X_1 + X_2 + \cdots + X_n)}\right]$$

$$= E\left[e^{j\omega X_1} \cdot e^{j\omega X_2} \cdot e^{j\omega X_3} \cdots e^{j\omega X_n}\right]$$

$$= E\left[e^{j\omega X_1}\right] \cdot E\left[e^{j\omega X_2}\right] \cdots E\left[e^{j\omega X_n}\right]$$

Since they are independent

$$= \left(e^{-\alpha(1 - e^{j\omega})}\right)^n = e^{-\alpha n}(1 - e^{j\omega})$$

S_n is a poisson RV with mean & Variance $= \alpha n = 2n$

(5) $P(370 < S_{200} < 460)$

$$= P\left(\frac{370-400}{\sqrt{400}} < \frac{S_{200} - 2\cdot 200}{\sqrt{400}} < \frac{460-400}{\sqrt{400}} \right)$$

$$= P\left(-\frac{3}{2} < \frac{S_{200} - 2\cdot 200}{\sqrt{2\cdot 200}} < 3 \right)$$

$$\approx \Phi(3) - \Phi(-1.5)$$

$$= \Phi(3) - (1 - \Phi(1.5))$$

$$\simeq 0.99865 + 0.9332 - 1$$

$$= 0.93185$$

Question 5: (30%) Consider a random process $X(n) = f_\zeta(n)$ where ζ is a binomial distribution with $n = 2$, $p = 1/3$. Namely, when $\zeta = 1$, $X(n) = f_1(n)$ and so on. We also know that $f_i(n)$ is a periodic function function period $i+1$, and $f_i(n)$ can be expressed by the following formula.

$$f_i(n) = \sum_{k=-\infty}^{\infty} \delta[n-(i+1)k].$$

For example, the values of $f_0(n)$, $f_1(n)$, and $f_2(n)$ for $n = 0, \cdots, 8$ are as follows.

$f_0(0) = 1, f_0(1) = 1, f_0(2) = 1, f_0(3) = 1, f_0(4) = 1, f_0(5) = 1, f_0(6) = 1, f_0(7) = 1, f_0(8) = 1.$
$f_1(0) = 1, f_1(1) = 0, f_1(2) = 1, f_1(3) = 0, f_1(4) = 1, f_1(5) = 0, f_1(6) = 1, f_1(7) = 0, f_1(8) = 1.$
$f_2(0) = 1, f_2(1) = 0, f_2(2) = 0, f_2(3) = 1, f_2(4) = 0, f_2(5) = 0, f_2(6) = 1, f_2(7) = 0, f_2(8) = 0.$

Answer the following questions.

1. (3%) What is the difference between a random variable and a random process.
2. (5%) What is the mean function $m_X(10)$.
3. (5%) What is the value of the auto-correlation function $R_X(2,5)$
4. (3%) Let $Y(n) = X(n) - X(n-1)$ be the output of a linear time-invariant system with $X(n)$ being the input. Find the mean function $m_Y(n)$ of Y in terms of the mean function $m_X(n)$ of X.
5. (4%) Construct the weight assignment of the random process $Y[n]$. Hint: There are two methods of constructing the weight assignment of a random process. Use the first method.
6. (4%) What is the definition of a "wide sense stationary" random process?
7. (6%) Is $X(n)$ a wide sense stationary random process? Why or why not?

1. R.P: The sample space is a collection of "functions".

R.V: The sample space - - - - - "values".

2. $P(\zeta=0) = \binom{2}{0}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = 1 \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P(\zeta=1) = \binom{2}{1}\left(\frac{1}{3}\right)^1 \times \left(\frac{2}{3}\right) = \frac{4}{9}$

$P(\zeta=2) = \binom{2}{2}\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$$m_X(10) = E(X(10)) = 1\times\left(\frac{4}{9}\right) + 1\times\left(\frac{4}{9}\right) + 0\times\frac{1}{9}$$

$$\because f_0(10)=1 \quad f_1(10)=1, \quad f_2(10)=0$$

3. $R_X(2,5) = E(X(2)X(5))$

$$= 1\times\frac{4}{9} + 0\times\frac{4}{9} + 0\times\frac{1}{9} = \frac{4}{9}$$

$$\because f_0(2)\cdot f_0(5) = 1$$
$$f_1(2)\times f_1(5) = 0$$
$$f_2(2)\times f_2(5) = 0$$

4. $m_Y(n) = E(Y(n))$

$$= E(X(n) - X(n-1))$$
$$= m_X(n) - m_X(n-1)$$

5. $$Y(n) = \begin{cases} f_0(n) - f_0(n-1) & \text{with prob } \frac{4}{9}. \\ f_1(n) - f_1(n-1) & \text{------ } \frac{4}{9} \\ f_2(n) - f_2(n-1) & \text{------ } \frac{1}{9} \end{cases}$$

6. W.S.S: The mean function is a constant.

& the auto-correlation function only depends on the time difference.

$E(X(0))=1$, $E(X(10))=\frac{8}{9}$

$\Rightarrow$ $m_x(n)$ is not constant

$\Rightarrow$ $X(n)$ is not W.S.S.