Question 1: (15%) A six-faced die, with faces numbered as $1, \cdots, 6$, is tossed and the number of dots facing up is noted.

1. (5%) Show that the event $A = \{X \geq 4\}$ "implies" $B = \{(X+4) \text{ is not a prime number}\}$ "implies" $C = \{X \geq 2\}$.
2. (5%) Construct the probability of the elementary events under the assumption that the face with a single dot is three times as likely to be facing up as any one of the rest 5 faces.
3. (5%) Under the above probability assignment, find the probabilities of $P(A)$, $P(B)$, $P(C)$.

1. $A = \{4, 5, 6\}$, $B = \{2, 4, 5, 6\}$, $C = \{2, 3, 4, 5, 6\}$

Therefore $A \subseteq B \subseteq C$.

2.

face #	1	2	3	4	5	6
Weight	$3x$	x	x	x	x	x

$\Rightarrow 3x + x + x + x + x + x = 8x = 1, \quad x = \frac{1}{8}$

Weight:	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

3. $P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

$P(B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$P(C) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}$

Question 2: (10%)

1. (3%) Using the Venn diagram to prove the DeMorgan's Rule: For all events $A, B \subseteq S$,

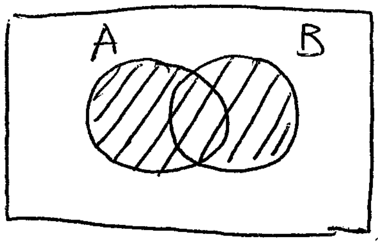
$$(A \cup B)^c = A^c \cap B^c.$$

2. (3%) Using the first result, prove

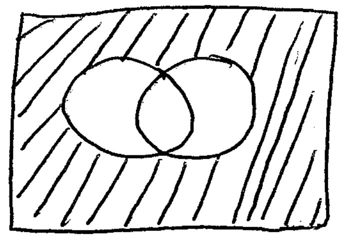
$$(A \cup B \cup C \cup D)^c = A^c \cap B^c \cap C^c \cap D^c.$$

3. (4%) Throw a fair die four times, and assume each die-tossing is independent. Compute the probability that a single dot is facing up at least once.

1.



$A \cup B$



$A^c \cap B^c$

2. first prove

$$(A \cup B \cup C)^c = (A \cup B)^c \cap C^c = A^c \cap B^c \cap C^c$$

then

$$(A \cup B \cup C \cup D)^c = (A \cup B \cup C)^c \cap D^c$$
$$= A^c \cap B^c \cap C^c \cap D^c$$

3. P(a single dot faces up at least once)

= 1 − P(a single dot NEVER faces up)

$$= 1 - \left(\frac{5}{6}\right)^4 = \frac{6^4 - 5^4}{6^4} = \frac{671}{1296}$$

Question 3: (7%) A random experiment has a sample space $S = \{x, y, z\}$. Suppose that $P(\{x, z\}) = 1/3$ and $P(\{y, z\}) = 7/9$. Use the axioms of probability to find the probabilities of the elementary events, namely, what are $P(\{x\})$, $P(\{y\})$, $P(\{z\})$.

We have

$$\begin{cases} P(\{x\}) + 0 + P(\{z\}) = \frac{1}{3} = P(\{x, z\}) \\ P(\{y\}) + P(\{z\}) = \frac{7}{9} = P(\{y, z\}) \\ P(\{x\}) + P(\{y\}) + P(\{z\}) = 1 = P(\{x, y, z\}) \\ \qquad = P(S) \end{cases}$$

Solve the simultaneous equations.

$$P(\{x\}) = \frac{2}{9}$$

$$P(\{y\}) = \frac{2}{3}$$

$$P(\{z\}) = \frac{1}{9}$$

Question 4: (12%) Two numbers X and Y are independently selected from the interval [0,1] uniformly randomly.

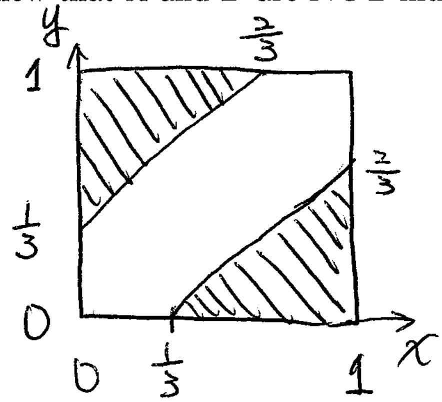
1. (5%) Find the probability that they differ by more than 1/3.

2. (7%) The two events A and B are defined as follows:

$$\begin{aligned} A &= \{X > 0.5\}, \\ B &= \{X > Y\}. \end{aligned}$$

Show that A and B are **NOT** independent.

1.



Sample space: $[0,1]\times[0,1]$

plot the region

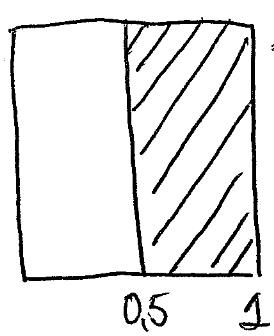
$\{|X-Y|>\frac{1}{3}\}$

$=\{X-Y>\frac{1}{3}\}\cup\{X-Y<-\frac{1}{3}\}$

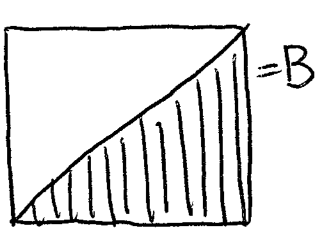
$P(\{|X-Y|>\frac{1}{3}\}) = \frac{\text{Area of}}{1}$

$=\frac{4}{9}$

2.

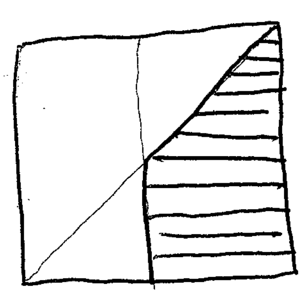


$=A$. $P(A)=0.5$



$=B$ $P(B)=0.5$

$A\cap B$:



$P(A\cap B)$

$=0.5\times1-\frac{1}{2}\times0.5\times0.5$

$=\frac{3}{8}$

$\because P(A\cap B)=\frac{3}{8}\neq P(A)\cdot P(B)$

$\therefore$ A & B are NOT independent

Question 5: (17%) A normal six-faced fair die is tossed and the number of dots N_1 is noted; an integer N_2 is then selected from $\{N_1, \cdots, 6\}$ uniformly randomly.

1. (2%) Specify the sample space.
2. (5%) Use either a tree diagram or a table method to construct the weight assignments (the probabilities).
3. (5%) Find the probability of the event $\{N_1 + N_2 \geq 10\}$.
4. (5%) Find the probability of the event $\{N_1 = 4\}$ given $\{N_2 = 6\}$.

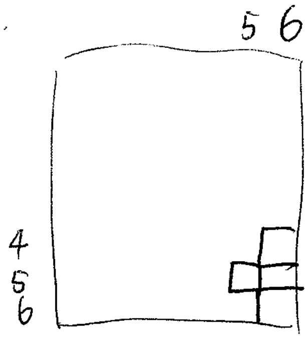
1.

N_1 \ N_2	1	2	3	4	5	6
1	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\frac{1}{6}$	$\frac{1}{6}\frac{1}{6}$
2		$\frac{1}{5}\times\frac{1}{6}$	$\frac{1}{5}\times\frac{1}{6}$	$\frac{1}{5}\times\frac{1}{6}$	$\frac{1}{5}\frac{1}{6}$	$\frac{1}{5}\frac{1}{6}$
3			$\frac{1}{4}\times\frac{1}{6}$	$\frac{1}{4}\times\frac{1}{6}$	$\frac{1}{4}\frac{1}{6}$	$\frac{1}{4}\times\frac{1}{6}$
4				$\frac{1}{3}\times\frac{1}{6}$	$\frac{1}{3}\frac{1}{6}$	$\frac{1}{3}\times\frac{1}{6}$
5					$\frac{1}{2}\times\frac{1}{6}$	$\frac{1}{2}\times\frac{1}{6}$
6						$1\times\frac{1}{6}$

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$

2. As in the above table

3.



Sum up the four blocks.

$$P(N_1+N_2 \geq 10) = \frac{1}{3}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{6} + 1\times\frac{1}{6} = \frac{7}{18}$$

4. $$P(N_1=4 \mid N_2=6) = \frac{P(N_1=4 \cap N_2=6)}{P(N_2=6)}$$

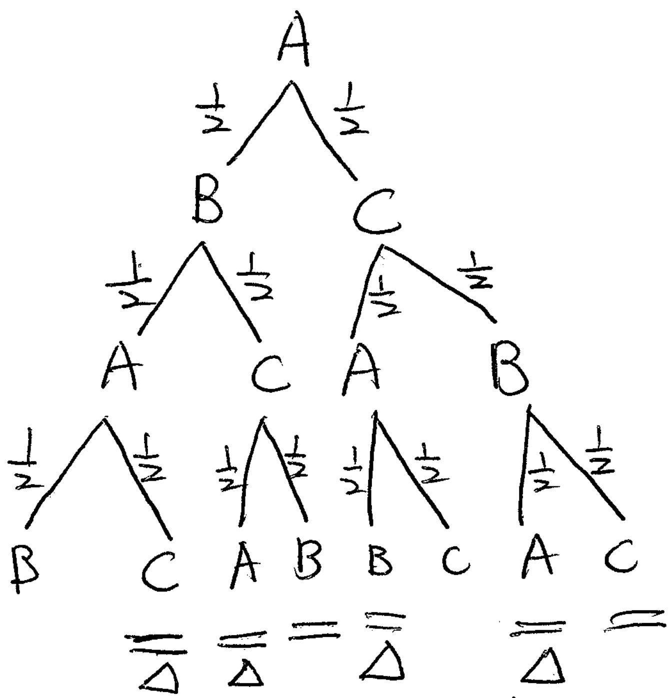
$$= \frac{\frac{1}{3}\times\frac{1}{6}}{\frac{1}{6}\times\frac{1}{6} + \frac{1}{5}\times\frac{1}{6} + \frac{1}{4}\times\frac{1}{6} + \frac{1}{3}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{6} + 1\times\frac{1}{6}}$$

$$= \frac{20}{147} = \frac{20}{147}$$

Question 6: (15%) A salesperson travelled between cities A, B, and C for four consecutive nights, and each night he could only stay in one city. At day 1, the salesperson started from city A and stayed there for the first night. For the next three mornings, the salesperson uniformly randomly selected the next destination from the two cities excluding the city he stayed for the last night. For example, if he stayed in city B for the third night, the fourth night of his can only be in either city A or city C.

1. (8%) What is the probability that the salesperson was able to visit all three cities?
2. (7%) What is the probability that the salesperson visited city A twice, given that the salesperson had visited all three cities.

1.



$P(\text{visit all cities}) = 6\times\left(\frac{1}{2}\right)^3 = \frac{3}{4}$ ···· all probabilites under which there is a "=" sign.

2.

$P(A \text{ is visited twice } \& \text{ all cities are visited})$

$= 4\times\left(\frac{1}{2}\right)^3 = \frac{1}{2}$ ···· all prob. with a "Δ" sign

$P(A \text{ is visited twice} \mid \text{all cities are visited}) = \frac{\frac{1}{2}}{\frac{3}{4}}$

$= \frac{2}{3}$

Question 7: (24%) The waiting time X of a customer in a queueing system is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy. The probabilities that he finds the system busy or idle are q and $1-q$, respectively.

1. (2%) What is the sample space? Hint, the waiting time t takes values in real numbers.
2. (7%) What is the corresponding cumulative distribution function (cdf)? Hint: consider two cases: $x < 0$ and $x \geq 0$.
3. (3%) Using the cdf obtained in 2., compute the probability $P(X \in [0,3])$?
4. (2%) What kind of random variables is X? A discrete random variable, a continuous random variable, or a mixed random variable? It is a multiple-choice question. No need to write down the justification.
5. (10%) Construct the generalized probability density function (pdf) using delta functions, namely, using $\delta(x - x_0)$.

   For your reference, the pdf of an exponential distribution is $f(t) = \lambda e^{-\lambda t}$, where $\lambda > 0$ is a parameter.

1. $\{\text{idle}, \text{busy}\} \times \mathbb{R}$

   or you can write $\{\text{idle}, \text{busy}\} \times [0, \infty)$

2. $$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ q \times \int_0^x \lambda e^{-\lambda t} dt + (1-q) \times 1 = q(1-e^{-\lambda x}) + (1-q) & \text{if } x \geq 0 \end{cases}$$

   the first term corresponds to the case that the system is busy. The 2nd term is the case that the system is idle

3. $$P(X \in [0,3]) = F_X(3) - F_X(0^-) = 1 - q e^{-3\lambda}$$

4. Mixed. Since there is a jump between $F_X(0^-) = 0$ & $F_X(0) = 1-q$

5. Identify the jump & do differentiation.

   $$f_X(x) = \delta(x) \cdot (1-q) + \begin{cases} q \times \lambda \times e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Question 8: (15%) Suppose X is a random variable uniformly distributed on $[-1, 1]$, and Y is defined by $Y = \frac{X^2}{2} + 1$.

1. (8%) Find the cumulative distribution function $F_Y(y) = P(Y \leq y)$. Hint: consider two cases: $y < 1$ and $y \geq 1$.

2. (7%) Find the probability density function $f_Y(y)$.

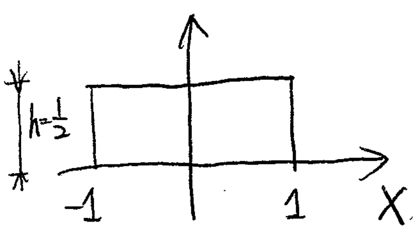
1. when $y < 1$. since $Y = \frac{X^2}{2} + 1 \geq 1$.

$F_Y(y) = P(Y \leq y) = 0$

when $y \geq 1$.

$$F_Y(y) = P(Y \leq y)$$

$$= P\left(\frac{X^2}{2} + 1 \leq y\right)$$



$$= P\left(-\sqrt{2(y-1)} \leq X \leq \sqrt{2(y-1)}\right)$$

$$= \begin{cases} \frac{1}{2} \times 2\sqrt{2(y-1)} & \text{if } \sqrt{2(y-1)} \leq 1 \\ 1 & \text{O.W. (otherwise)} \end{cases}$$

$$= \begin{cases} \sqrt{2(y-1)} & \text{if } y \leq \frac{3}{2} \\ 1 & \text{Otherwise} \end{cases}$$

2. Simply do differentiation since there is no jump.

$$f_X(y) = \begin{cases} 0 & \text{if } y < 1. \\ \frac{1}{2} \frac{2}{\sqrt{2(y-1)}} & \text{if } 1 \leq y \leq \frac{3}{2} \\ 0 & \text{if } \frac{3}{2} < y \end{cases}$$