ECE 302, Final

3:20-5:20pm Mon. May 1, WTHR 160 or WTHR 172.

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains 7 questions and is worth 100 points + 18 bonus points. You have two hours to complete it. So basically, you only need to answer $\approx$5–6 questions correctly to get full grade. I will suggest not spending too much time on a single question, and work on those you know how to solve.
4. The sub-questions of a given question are listed from the easiest to the hardest. The best strategy may be to finish only the sub-questions you know exactly how to solve.
5. Write down as much your knowledge about how to solve the problem. You might get some partial credit.
6. There are a total of 14 pages in the exam booklet. Use the back of each page for rough work.
7. **Neither calculators nor help sheets are allowed.**
8. Read through all of the problems first, and consult with the TA during the first 20 minutes. After that, no questions should be asked unless under special circumstances, which is at TA's discretion. You can also get a feel for how long each question might take after browsing through the entire question set. Good luck!

Name:

Student ID:

E-mail:

Signature:

Bernoulli distribution with parameter p:

$$\begin{aligned} \mathsf{P}(X=0) &= 1-p, \qquad \mathsf{P}(X=1)=p \\ \mathsf{E}(X) &= p \\ \mathsf{Var}(X) &= p(1-p). \end{aligned}$$

Poisson random variable (Poisson distribution) with parameter λ:

$$\begin{aligned} \mathsf{P}(X=k) &= \frac{\lambda^k}{k!}e^{-\lambda}, \quad \forall k \in \{0,1,\cdots,n,\cdots\} \\ \mathsf{E}(X) &= \lambda \\ \mathsf{Var}(X) &= \lambda. \end{aligned}$$

Geometric random variable (geometric distribution) with parameter p:

$$\begin{aligned} \mathsf{P}(X=k) &= p(1-p)^k, \quad \forall k \in \{0,1,\cdots,n,\cdots\} \\ \mathsf{E}(X) &= \frac{1-p}{p} \\ \mathsf{Var}(X) &= \frac{1-p}{p^2}. \end{aligned}$$

Gaussian random variable (Gaussian distribution) with parameters (m, σ^2):

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-m)^2}{2\sigma^2}} \\ \mathsf{E}(X) &= m \\ \mathsf{Var}(X) &= \sigma^2. \end{aligned}$$

Discrete Fourier transform:

$$S(f) = \mathcal{F}(R(n)) = \sum_{n=-\infty}^{\infty} R(n)e^{-2\pi fn}.$$

Continuous Fourier transform:

$$S(f) = \mathcal{F}(R(t)) = \int_{t=-\infty}^{\infty} R(t)e^{-2\pi ft}.$$

Question 1: (20%) Suppose X is the result of tossing a fair die with 6 faces, namely, $\mathsf{P}(X = i) = 1/6$ for $i = 1, \cdots, 6$. Another computer program independently generates a random number Y. Prof. Wang assumes that Y is generated by the waiting time until the first head shows up during flipping a fair coin. Let $Z = X + Y$ being a new random variable.

1. (1%) What type of distribution is Y?
2. (2%) Find the sample space of Z.
3. (3%) Find the mean value $\mathsf{E}(Z)$
4. (2%) Find the variance $\mathsf{Var}(Z)$.
5. (4%) Find the probability that $\mathsf{P}(Z = 2)$. Hint: Count the possible outcomes in the equivalent event. That is, identify the situations in which we will have $Z = 2$.
6. (4%) Find the conditional probability that $\mathsf{P}(X = 1|Z = 2)$. If you do not know how to answer this question, write down the definition/meaning of the conditional probability, which will give you half the credit.
7. (4%) After looking really into the computer software, TA tells Prof. Wang that Y is actually a Poisson random variable with parameter $\lambda = 1$. Does it affect your answers of sub-questions 1, 2, 3, 4? It is actually 4 yes/no questions. Your answer should be something like (Yes, Yes, Yes, No), plus four sentences to explain each of your results.

1. Y is a geometric distribution

2. The sample space of Z is

$\{1, 2, 3, 4, \cdots, n, \cdots\}$. Namely, all positive integers

3. $E(Z) = E(X) + E(Y)$

$= \left(\frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \frac{1}{6}\times 4 + \frac{1}{6}\times 5 + \frac{1}{6}\times 6\right) + \frac{1-0.5}{0.5}$

$= 4.5$

4. $Var(Z) = Var(X) + Var(Y)$

in which $Var(X) = E(X^2) - (E(X))^2$

$= \left(\frac{1}{6}\times 1 + \frac{1}{6}\times 4 + \frac{1}{6}\times 9 + \frac{1}{6}\times 16 + \frac{1}{6}\times 25 + \frac{1}{6}\times 36\right) - 3.5^2$

$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

$$Var(Y) = \frac{1-0.5}{0.5^2} = 2$$

$$\Rightarrow Var(Z) = \frac{59}{12}.$$

5. $P(Z=2) = P(X=1, Y=1) + P(X=2, Y=0)$

$$= \frac{1}{6} \times 0.5(1-0.5)^1 + \frac{1}{6} \times 0.5(1-0.5)^0$$

$$= \frac{1}{8}$$

6. $P(X=1 \mid Z=2) = \frac{P(X=1, Z=2)}{P(Z=2)}$

$$= \frac{P(X=1, Y=1)}{P(Z=2)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{4}}{\frac{1}{8}} = \frac{1}{3}$$

7. ~~(N (Yes)~~ [illegible]

Yes: Y becomes a Poisson

No: The sample space is the same.

No: $E(Y_{new}) = \lambda = 1 = E(Y_{old}) = \frac{1-p}{p} = 1.$

Yes: $Var(Y_{new}) = \lambda = 1 \neq Var(Y_{old}) = \frac{1-p}{p^2} = 2$

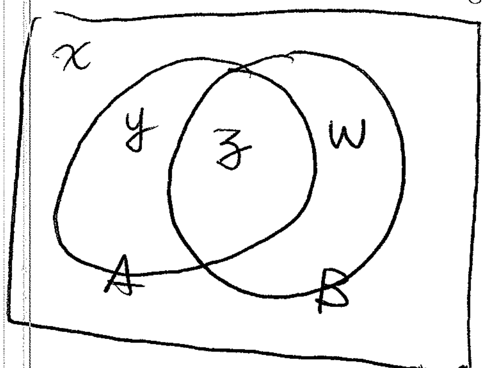
Question 2: (10%) Suppose we have two events A and B, and we know $P(A) = 0.4$ $P(B) = 0.7$ and $P(A \cup B) = 0.9$.

1. (9%) Find $P(A \cap B^c)$. (B^c is the compliment of B, namely, B^c contains everything not in B.)

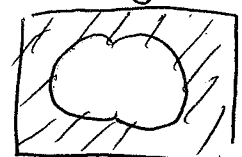
2. (1%) Are A and B independent? Use one sentence to justify your answer.

Hint: Use the Venn diagram and find out the weight assigned to each region.

1



Let x denote the weight on region



y, z, w denote the weights similarly.

By the statements of the question, we have.

$$\begin{cases} x+y+z+w=1 \\ y+z=0.4 \\ z+w=0.7 \\ y+z+w=0.9 \end{cases} \Rightarrow \begin{cases} x=0.1 \\ y=0.2 \\ z=0.2 \\ w=0.5 \end{cases}$$

$$P(A \cap B^c) = y = 0.2$$

2. No: $P(A \cap B) = z = 0.2 \neq P(A) \times P(B)$

$= 0.4 \times 0.7$

Question 3: (10%) Suppose X is Gaussian distributed with mean m_X and variance σ_X^2, and Y is Gaussian distributed with mean m_Y and variance σ_Y^2. Suppose the correlation coefficient is ρ.

1. (4%) Find the covariance between X and Y.
2. (6%) Let $Z = X + Y$. Find the probability density function of Z. (If you do not know the answer of the previous question, you can use the covariance $\mathsf{Cov}(X, Y)$ as part of your answer.)

1\.

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \Rightarrow Cov(X,Y) = \rho \times \sigma_X \times \sigma_Y$$

2\.

$$E(Z) = E(X) + E(Y) = m_X + m_Y$$

$$Var(Z) = Var(X) + Var(Y) + 2Cov(X,Y)$$

$$= \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y$$

$$f(z) = \frac{1}{\sqrt{2\pi(\sigma_X^2+\sigma_Y^2+2\rho\sigma_X\sigma_Y)}} \times e^{-\frac{(z-m_X-m_Y)^2}{2(\sigma_X^2+\sigma_Y^2+2\rho\sigma_X\sigma_Y)}}$$

Question 4: (10%) Let $X(t)$ be a random process.

1. (3%) Write down the definition of the mean function $m_X(t)$ and the auto correlation function $R_X(t_1, t_2)$.
2. (3%) If $Y(t) = 3X(t) + 2$, find out the mean $m_Y(t)$ and $R_Y(t_1, t_2)$ in terms of $m_X(t)$ and $R_X(t_1, t_2)$.
3. (4%) Suppose $X(t)$ is an independently and identically distributed Bernoulli random process with parameter p. Note in this question, p may not be $1/2$. Find the probability $\mathsf{P}(X_1 = 0, X_{17} = 1, X_{101} = 1)$.

1. $m_X(t) = E(X(t))$

$R_X(t_1, t_2) = E(X(t_1)X(t_2))$

2. $m_Y(t) = E(Y(t)) = E(3X(t)+2)$

$= 3m_X(t) + 2$

$R_Y(t_1, t_2) = E(Y(t_1)Y(t_2))$

$= E((3X(t_1)+2)(3X(t_2)+2))$

$= 9E(X(t_1)X(t_2)) + 6E(X(t_1)) + 6E(X(t_2)) + 4$

$= 9R_X(t_1, t_2) + 6m_X(t_1) + 6m_X(t_2) + 4$

3. Since it is i.i.d. Bernoulli

$\Rightarrow P(X_1 = 0, X_{17} = 1, X_{101} = 1)$

$= P(X_1 = 0) \cdot P(X_{17} = 1) \cdot P(X_{101} = 1)$

$= (1-p) \times p \times p = (1-p)p^2$

Question 5: (15%) The joint pdf of a 2-D random variable (X, Y) is

$$f_{XY}(x,y) = \begin{cases} ce^{-x-y} & \text{if } 0 \le x \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

1. (3%) One student said that c must be 2, which is correct. If you were that student, how would you solve the value of c. Use one sentence and one formula to explain what you will do. If there are integrals involved, explicitly specify the range of the interval.

2. (3%) That student also correctly solved part of the cumulative distribution function of (X, Y) as

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-2x} - 2e^{-y}(1-e^{-x}) & \text{if } 0 \le x \le y < \infty \\ g_1(x,y) & \text{if } 0 \le y \le x < \infty \\ g_2(x,y) & \text{if either } x < 0 \text{ or } y < 0 \end{cases} \quad (1)$$

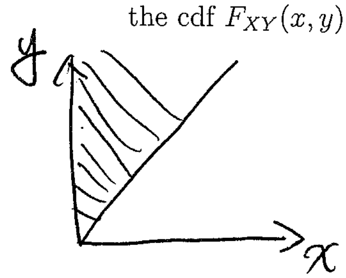
   Find $g_2(x,y)$ and explain your result.

3. (3%) Find $g_1(x,y)$. Hint: There are two methods: direct integration or reuse some given results.

4. (3%) Find the probability $\mathsf{P}(2 < X \le 5, 3 < Y \le 6)$ in terms of $F_{XY}(x,y)$.

5. (3%) Evaluate the value of your previous question by really substituting values into the cdf $F_{XY}(x,y)$.

1.

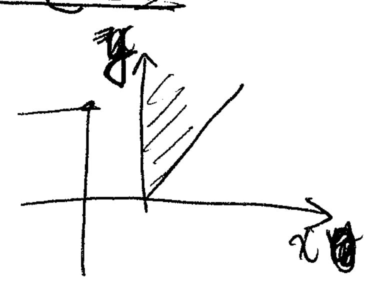


The total ~~area~~ weight must be 1.
So that the following equation must hold.

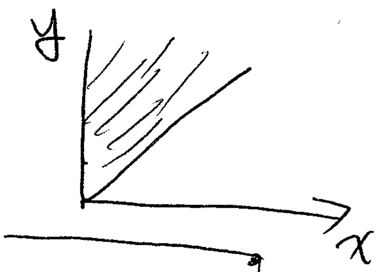
$$\int_0^\infty \int_0^y ce^{-x-y}\,dx\,dy = 1. \quad \#$$

2. $g_2(x,y) = 0$

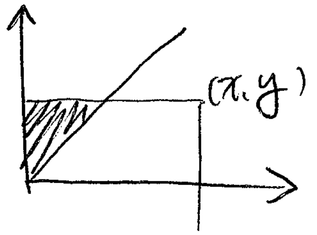
$\because$ either



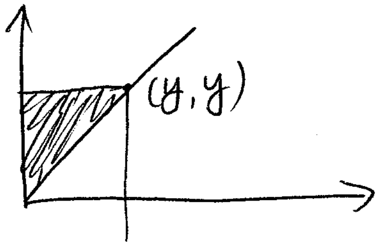
or



3.



we would like to find the volume above the triangle area which is the same as.



By substituting (y, y) to the case that $0 \le x \le y < \infty$

we have $g_1(x, y)$

$$= 1 - e^{-2y} - 2e^{-y}(1 - e^{-y})$$

$$= 1 - 2e^{-y} + e^{-2y}$$

3. $P(2 < X \le 5, 3 < Y \le 6)$

$$= F_{XY}(5,6) - F_{XY}(2,6) - F_{XY}(5,3) + F_{XY}(2,3)$$

4. $F_{XY}(5,6) = 1 - e^{-10} - 2e^{-6}(1 - e^{-5})$

$F_{XY}(2,6) = 1 - e^{-4} - 2e^{-6}(1 - e^{-2})$

$F_{XY}(5,3) = 1 - 2e^{-3} + e^{-6}$

$F_{XY}(2,3) = 1 - e^{-4} - 2e^{-3}(1 - e^{-2})$

$\Rightarrow P(2 < X \le 5, 3 < Y \le 6)$

$$= 2e^{-5} - 2e^{-8} - e^{-6} + 2e^{-11} - e^{-10}$$

Question 6: (36%)

A basketball game involving teams A and B is simulated by two computers indefinitely. Namely, at time $n = 1$, there are two virtual games simulated by computers 1 and 2, and at time $n = 2$, there are two *new* virtual games simulated by computers 1 and 2, and so on so forth. We say the result of each virtual game is 1 if A wins and -1 if B wins. So we have two random processes $X_1(n)$ and $X_2(n)$ denoting the results of virtual games from both computers, each of which takes values in $\{-1, 1\}$.

Suppose the software engineer designed the simulation software of computer 1 as follows. Randomly flip one and only one fair coin. If the result is a head, then $X_1(n) = 1$ for all n. Otherwise, $X_1(n) = -1$ for all n. (Note: only 1 coin flipping is performed in computer 1 no matter how large n is.) The simulation software of computer 2 is designed differently. For every n, a new fair coin is flipped, and if the outcome is head, $X_2(n) = 1$, otherwise $X_2(n) = -1$. For the next time instance, another fair coin is flipped and $X_2(n+1)$ is determined accordingly.

1. (4%) Find the mean function $m_{X_1}(n)$ and the auto-correlation function $R_{X_1}(n_1, n_2)$.
2. (1%) Is $X_1(n)$ a wide-sense stationary random process? Use one sentence to explain your result.
3. (4%) Find the mean function $m_{X_2}(n)$ and the auto-correlation function $R_{X_2}(n_1, n_2)$.
4. (1%) Is $X_2(n)$ a wide-sense stationary random process? Use one sentence to explain your result.
5. (3%) A naive prediction system is to use moving average. To be more explicitly, $Y(n) = \frac{1}{2}(X(n-1) + X(n-2))$. Find the impulse response $h(t)$.
6. (5%) Find the frequency response $H(f)$. Find the power spectral density functions of X_1 and X_2. (Namely, find $S_{X_1}(f)$ and $S_{X_2}(f)$.)
7. (1%) Is $X_1(n)$ white? Is $X_2(n)$ white? Use one sentence to explain your result.
8. (5%) Find m_Y, $S_Y(f)$ and $S_{X_1Y}(f)$, when this predictor is applied to $X_1(n)$.
9. (4%) Find $R_Y(n)$ and $R_{X_1Y}(n)$, when this predictor is applied to $X_1(n)$.
10. (2%) Find m_Y, $S_Y(f)$ and $S_{X_2Y}(f)$, when this predictor is applied to $X_2(n)$.
11. (1%) Find $R_Y(n)$ and $R_{X_2Y}(n)$, when this predictor is applied to $X_2(n)$.
12. (5%) Suppose you would like to bet your money on these virtual games based on this moving average predictor. On which computer would you like to apply this $Y(n)$ predictor? Choose your answer among (a) apply it to $X_1(n)$, (b) apply it to $X_2(n)$, or (c) it does not make any difference between the two. Explain your decision.

1. $m_{x_1}(n) = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0.$

$R_{x_1}(n_1, n_2) = E(X_1(n_1) X_1(n_2))$

$= \frac{1}{2} \times (1 \times 1) + \frac{1}{2} \times (-1) \times (-1) = 1.$

2. Yes: since $m_{x_1}(n) = 0.$ and $R_{x_1}(n_1, n_2) = 1.$

3. $m_{x_2}(n) = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0.$

$R_{x_2}(n_1, n_2) = E(X_2(n_1) X_2(n_2))$

if $n_1 \neq n_2 \Rightarrow \frac{1}{4} 1 \times 1 + \frac{1}{4} \times 1 \times (-1) + \frac{1}{4} \times (-1) \times 1)$

$+ \frac{1}{4}(-1)(-1) = 0$

if $n_1 = n_2 \Rightarrow \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times (-1) \times (-1) = 1$

4. Yes: since $m_{x_2}(n) = 0.$ and $R_{x_2}(n_1, n_2)$ only depends on $n_2 - n_1$

5. Impulse response: $\cancel{\frac{1}{2}\delta(n) + \frac{1}{2}}$

$\frac{1}{2}\delta(n-1) + \frac{1}{2}\delta(n-2))$

6. $H(f) = \mathcal{F}(h(n))$

$= \frac{1}{2} e^{-j2\pi f} + \frac{1}{2} e^{-j2\pi \times 2f}$

$S_{x_1}(f) = \mathcal{F}(R_{x_1}(n)) = \mathcal{F}(1) = \delta(f))$

$S_{x_2}(f) = \mathcal{F}(R_{x_2}(n)) = \mathcal{F}(\delta(n)) = 1.$

7. $X_1(n)$ is not white since $S_{X_1}(f)$ is not flat.

$X_2(n)$ is white since $S_{X_2}(f)$ is flat.

8. $m_Y = H(0)\, m_{X_1}$

$= 0.$

$S_Y(f) = |H(f)|^2 S_X(f)$

$= (2 + 2\cos(2\pi f)) \times \frac{1}{4} \times \delta(f)$

$= \delta(f)$

$S_{X_1Y}(f) = H(f)^* S_X(f) = \left(\frac{1}{2} e^{+j2\pi f} + \frac{1}{2} e^{+j2\pi 2f}\right) \delta(f)$

$= \delta(f)$

9. $R_Y(n) = \mathcal{F}^{-1}(\delta(f)) = 1.$

$R_{XY}(n) = \mathcal{F}^{-1}(\delta(f)) = 1.$

10. $m_Y = H(0)\, m_X = 0.$

$S_Y(f) = \frac{1}{2}(1 + \cos(2\pi f)) \times 1 = \frac{1}{2}(1 + \cos(2\pi f))$

$= |H(f)|^2 S_{X_2}(f)$

$S_{X_2Y}(f) = H(f)^* S_{X_2}(f) = \left(\frac{1}{2} e^{+j2\pi f} + \frac{1}{2} e^{j4\pi f}\right) \times 1$

$= \left(\frac{1}{2} e^{j2\pi f} + \frac{1}{2} e^{j4\pi f}\right)$

11. $R_Y(n)$ $\cancel{R_Y(f)} = \mathcal{F}^{-1}\left(\frac{1}{2}(1 + \cos(2\pi f))\right) = \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{4}\delta(n+1)$

$$R_{X_2Y}(n) = \mathcal{F}^{-1}\left(\frac{1}{2}e^{j2\pi f} + \frac{1}{2}e^{j4\pi f}\right)$$

$$= \frac{1}{2}\delta(n+1) + \frac{1}{2}\delta(n+2)$$

12. I would apply $Y(n)$ to $X_1(n)$

$\because$ $E(X_1(n)Y(n))$

$$= R_{X_1Y}(0) = 1 > E(X_2(n)Y(n))$$

$$= R_{X_2Y}(0) = 0$$

This is saying $X_1(n)$ & $Y(n)$ are "highly correlated". So betting on the result of $Y(n)$ is very profittable.

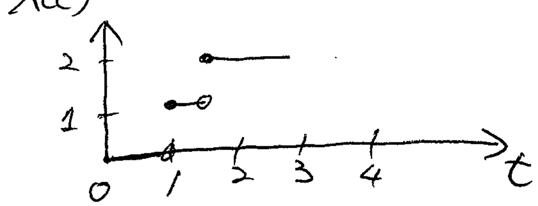
Question 7: (17%) The arrival of customers is modelled by a Poisson random process, $X(t)$, where t takes value from any non-negative real numbers, and $X(t)$ takes value in $\{0, 1, \cdots, n \cdots\}$. The parameter of the Poisson random process is $\lambda = 1$. (You can use either λ or 1 for your convenience.)

1. (2%) Suppose the first customer arrives at $t = 1.0$, and the second customer arrives at $t = 1.5$. Plot a sample path/a realization corresponds to this description.
2. (3%) Suppose the "fifth" customer arrives at $t = 3.25$. Plot one sample path compatible with this description. Hint: There may be more than one such path.
3. (4%) Find the probability that no customer arrives between $[0, 5]$ and $[7, 10]$.

=== Do the following questions if you have time ===

4. (4%) Find the probability that there are equal number of customers arrived in the first three intervals $(0, 1]$, $(1, 2]$ and $(2, 3]$. Hint: consider different sub-cases that there are exactly k, k, and k customers arrived in $(0, 1]$, $(1, 2]$ and $(2, 3]$.
5. (4%) Find the probability that there are 2 customers in $[0, 2]$ and 2 customers in $[1, 3]$. Hint: The independent increment property only works when the underlying intervals are disjoint. How to convert $[0, 2]$ and $[1, 3]$ into disjoint intervals?

1.



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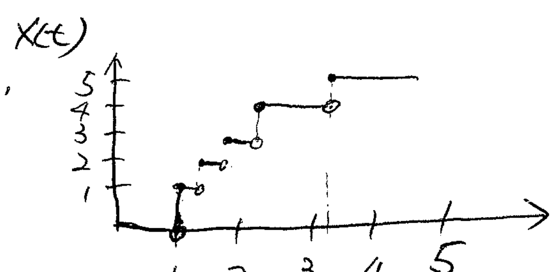

3. P(no customer between [0,5]) × P(no customer between [7,10])

$$= \frac{(\lambda \times 5)^0}{0!} e^{-5\lambda} \times \frac{(3 \times \lambda)^0}{0!} e^{-3\lambda} = e^{-8\lambda}$$

4. $\sum_{k=0}^{\infty} P(k \text{ in } (0,1], k \text{ in } (1,2], k \text{ in } (2,3])$

$= \sum_{k=0}^{\infty} P(k \text{ in } (0,1]) \cdot P(k \text{ in } (1,2]) \cdot P(k \text{ in } (2,3])$

$$= \sum_{k=0}^{\infty} \left(\frac{\lambda^k}{k!} e^{-\lambda} \right)^3$$

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5. Divide $[0,2]$ & $[1,3]$ into 3 disjoint intervals

$[0,1)$, $[1,2]$, $(2,3]$,

there are three different cases.

$$P(2 \text{ in } [0,1), 0 \text{ in } [1,2], 2 \text{ in } (2,3])$$

$$+P(1 \text{ in } [0,1), 1 \text{ in } [1,2], 1 \text{ in } (2,3])$$

$$+P(0 \text{ in } [0,1), 2 \text{ in } [1,2], 0 \text{ in } (2,3])$$

$$= \frac{\lambda^2}{2!} e^{-\lambda} \cdot \frac{\lambda^0}{0!} e^{-\lambda} \cdot \frac{\lambda^2}{2!} e^{-\lambda}$$

$$+ \frac{\lambda^1}{1!} e^{-\lambda} \cdot \frac{\lambda^1}{1!} e^{-\lambda} \cdot \frac{\lambda^1}{1!} e^{-\lambda}$$

$$+ \frac{\lambda^0}{0!} e^{-\lambda} \cdot \frac{\lambda^2}{2!} e^{-\lambda} \frac{\lambda^0}{0!} e^{-\lambda}$$

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