* Average of indep. identically distributed (i.i.d.) $X_1, X_2, \cdots, X_n$ R.V.

Ex: throw a dice 1000 times.

pull a slot machine 10000 times.

Suppose each X_i has a common mean μ_X and common variance σ_X^2

the $\boxed{\text{sample mean}}$ is defined

as $M_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

n is termed the $\boxed{\text{sample size}}$

Note that a "sample mean" is not a constant but a Random variable

$$E(M_n) = \frac{1}{n}(\mu_X + \mu_X + \cdots + \mu_X)$$

$$= \mu_X.$$

$$Var(M_n) = \frac{1}{n^2}(\sigma_X^2 + \sigma_X^2 + \cdots + \sigma_X^2)$$

$$= \frac{1}{n}\sigma_X^2 \quad \left(\begin{array}{l}\text{No cross-terms} \\ \because \text{independence}\end{array}\right)$$

The $\boxed{\text{sample mean}}$ has the same mean of X but with reduced variance.

When the variance approaches zero, the sample mean M_n concentrates around the (actual) mean

Chebyshev Inequality

$$P(|M_n - \mu| \geq \varepsilon) \leq \frac{Var(M_n)}{\varepsilon^2} = \frac{\sigma_x^2}{\varepsilon^2 \times n}$$

☆ Weak Law of Large Numbers. (WLLN)

$$\lim_{n \to \infty} P(|M_n - \mu| \geq \varepsilon) = 0$$

for any ε.

Note: WLLN applies to any (marginal) distribution X.

Intuition:

Set a large number $n = 100,000$. Even before I throw the dice n times, I am confident that once I finish throwing the dice, with high prob my "sample mean" (the avg of the outcome) will be very close to the true mean μ.

* Strong Law of Large Numbers

$$P\left(\lim_{n\to\infty} M_n = \mu\right) = 1$$

Intuition:

① Fix $n = 100{,}000$, WLLN says that with high prob $M_{100,000}$ is close to μ. However with $\boxed{\text{small prob}}$, $M_{100,000}$ may be far from μ. ② The SLLN says that even if I had a bad luck in the first 100,000 throws, as long as I am persistent and keep throwing the dice, in the end the <u>sample mean</u> will come back to μ. (Although we may need to throw the dice 1 million more times.)

* WLLN & SLLN are used to model the system

Noise Voltage $\longrightarrow$ modeled as Gsn with $\mu_x = 0$, $\sigma_x = 1$.

→ Sample it → check the sample mean update the μ

$M_n = \frac{1}{n}(X_1 + \cdots + X_n)$, $W_n = \frac{1}{n}(X_1^2 + X_2^2 + \cdots + X_n^2)$

* Central Limit Theorem (CLT)
- applicable to <u>any</u> distribution

$S_n = X_1 + \cdots + X_n$ the sum of i.i.d R.V.s.

$\mu_S = n\mu_X$

$\sigma_S^2 = n\sigma_X^2$ $\sigma_S = \sqrt{n}\,\sigma_X$

Q: How to <u>nomalize</u> S_n by a linear operation $Z_n = aS_n + b$ such that Z_n has zero mean & variance 1

Ans: $Z_n = \frac{1}{\sigma_S}(S_n - \mu_S)$

$$= \frac{S_n - n\mu_X}{\sqrt{n}\cdot\sigma}$$

Q: What is the distribution of Z_n?

Ans: Without the knowledge about the distri

~~of~~ X_i, I do not know the exact

distribution of Z_n. but the central limit theorem guarantees that it is going to be "close" to a standard Gsn. Moreover, the larger the sample size n is, the closer the distribution of Z_n is to a standard Gsn.

Central Limit Theorem

$$\lim_{n\to\infty} P(Z_n \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

3 Main Ingredients:

① S_n: Sum of i.i.d $X_1, \cdots, X_n$.

② Normalize $S_n \longrightarrow Z_n$

③ Z_n has a distribution close to that of a $\boxed{\text{standard Gsn}}$

Note: Generally, the prob arguments are sensitive to the underly Weight Assignment. However WLLN, SLLN, CLT show that sometimes different W.A may have the same properties.

* Using the CLT.

Application I: Gaussian Approximation for other R.Vs.

Ex: X_i : the outcome of an unfair dice

$$P_1 = \frac{1}{21},\ P_2 = \frac{2}{21},\ \cdots,\ P_6 = \frac{6}{21}$$

$$S_{100} = X_1 + \cdots + X_{100}$$

Q: $P(S_{100} < 400.5) = ?$

Ans: Solution 1: Step 1: Find the pmf of S_{100}. (the sample space of S_{100} is $\{100, 101, \cdots, 600\}$)

by $\boxed{\text{convolution}}$

Step 2: Count: $P_{100} + P_{101} + \cdots + P_{400}$.

Solution #2:

Step 1: Normalize S_{100}

$$E(S_{100}) = 100\,E(X) = 100 \times \frac{91}{21}$$

$$Var(S_{100}) = 100\,Var(X) = 100 \times \frac{20}{9}$$

$$\sigma_{S_{100}} = \frac{10\sqrt{20}}{3} = \frac{20\sqrt{5}}{3}$$

$$Z_{100} = \frac{1}{\frac{20\sqrt{5}}{3}}\left(S_{100} - \frac{9100}{21}\right)$$

Step 2: $P(S_{100} < 400)$

$$= P\left(\frac{1}{\frac{20\sqrt{5}}{3}}\left(S_{100} - \frac{9100}{21}\right) < \frac{1}{\frac{20\sqrt{5}}{3}}\left(400 - \frac{9100}{21}\right)\right)$$

$$= P(Z_{100} < -2.236)$$

CLT

$$\approx P(Z < -2.236)$$

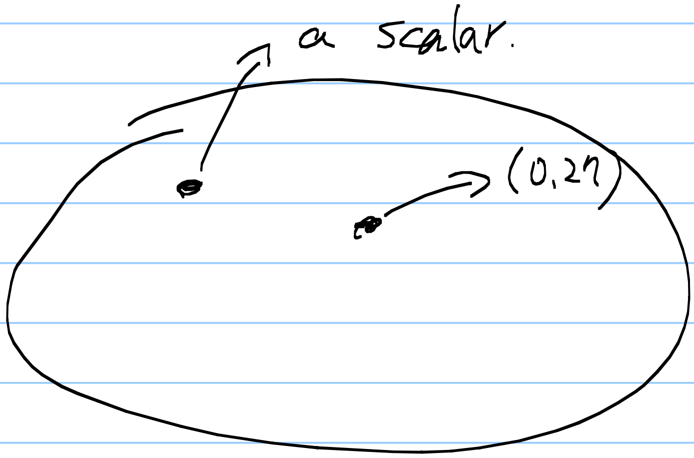
Z is standard Gsn

$$= \Phi(-2.236) = Q(2.236)$$

$$= 1.27\%$$

* Random Process

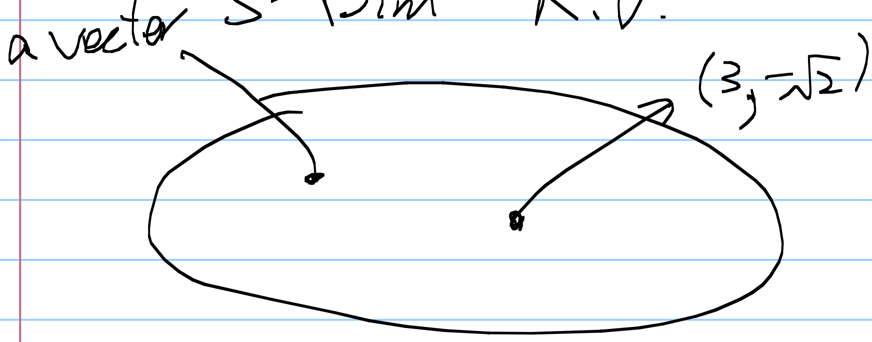
Review : 1-Dim R.V.



Sample space

W.A for each scalar.

3-Dim R.V.



Sample space

W.A for each vector.

Ex: $P(X=(0,0.5,1))=0.5$

$P(X=(1,-3,-0.5))=0.3$

$P(X=(\pi,\sqrt{3},\sqrt{2}))=0.2$

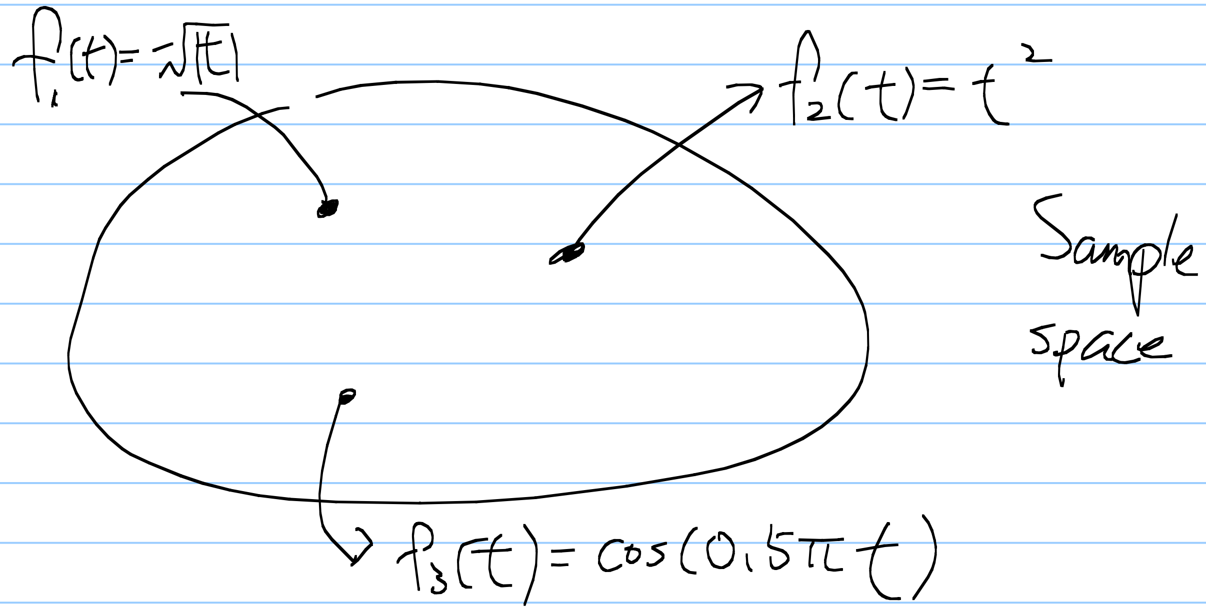
Q: P(the second coordinate of $X>0$)

Ans: $=0.5+0.2$

Q P(At least one coordinate of X is less than 0) = 0.3

Random processes

∞-dim R.V.



Example

$$P(X(t)=f_1(t))=0.5$$
$$P(X(t)=f_2(t))=0.3$$
$$P(X(t)=f_3(t))=0.2$$

Q: $P(X(4)>0)$	Q: $P(X(t)<0$ for some $t)$
Ans: $0.3+0.2=0.5$	Ans: $0.5+0.2=0.7$

We call the "random function $X(t)$" as a Random process.

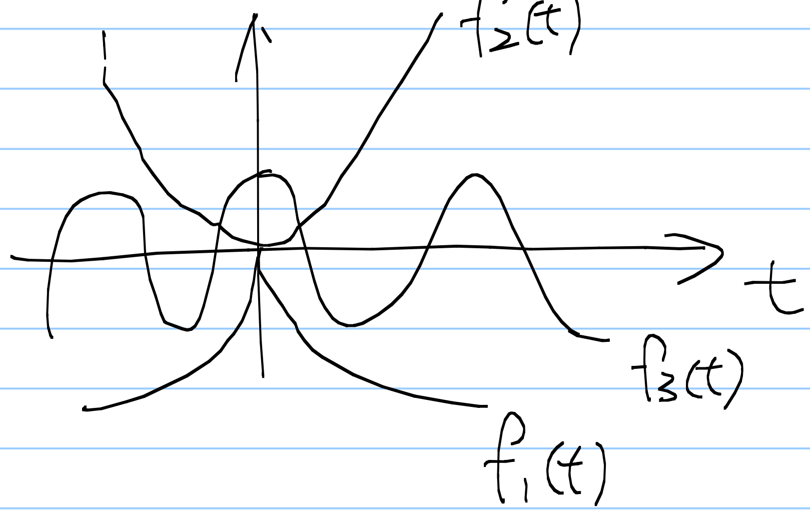
Remark: $X(t)$ is a Random Process

$X(1)$, $X(2.3)$, $X(-1)$ are R.V.s.

$(X(3), X(\pi))$ are joint R.V.

The outcome of a R.P $X(t)$ is a "function" $x(t)$, which is called the "sample path" (or "sample function" or "realization")

Example: there were three sample paths in the previous example

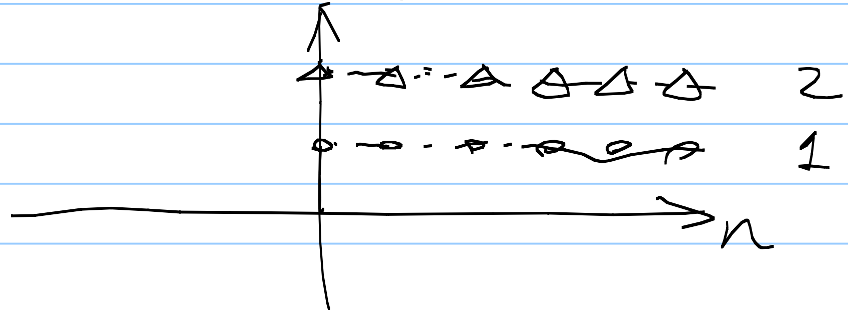


Ex: Prob 9.2

A fair die is tossed, & the output is k. Once k is decided, $X[n] = k$ for all n.

Q: Plot some sample path of $\underset{6}{\text{the}}$ R.P $X[n]$

Ans:



each with weight $\frac{1}{6}$

Q: Find the pmf of X_n (say X_{16})

Ans: X_{16} can ~~take~~ values in $\{1, \cdots, 6\}$
with prob $\frac{1}{6}, \frac{1}{6}, \cdots, \frac{1}{6}$

Q: Find the joint pmf of X_n, X_{n+k}
(say X_1, X_5)

Ans:

X_n \ X_{n+k}	1	2	3	4	5	6
1	$\frac{1}{6}$					
2		$\frac{1}{6}$				
3			$\frac{1}{6}$			
4				$\frac{1}{6}$		
5					$\frac{1}{6}$	
6						$\frac{1}{6}$

* Random Process $X(t)$ or $X[n]$

Sample space: $\{$ all functions $x(t)$ $(x[n])\}$

W.A: Two methods of describing the W.A.

Method 1: We write $X(t) = X(t, \zeta)$
where ζ represents the "Random part"
and we specify the W.A of ζ.

Ex: $X(t, \zeta) = \underline{\underline{\zeta}} \, \underline{\underline{\cos(2\pi t)}}$

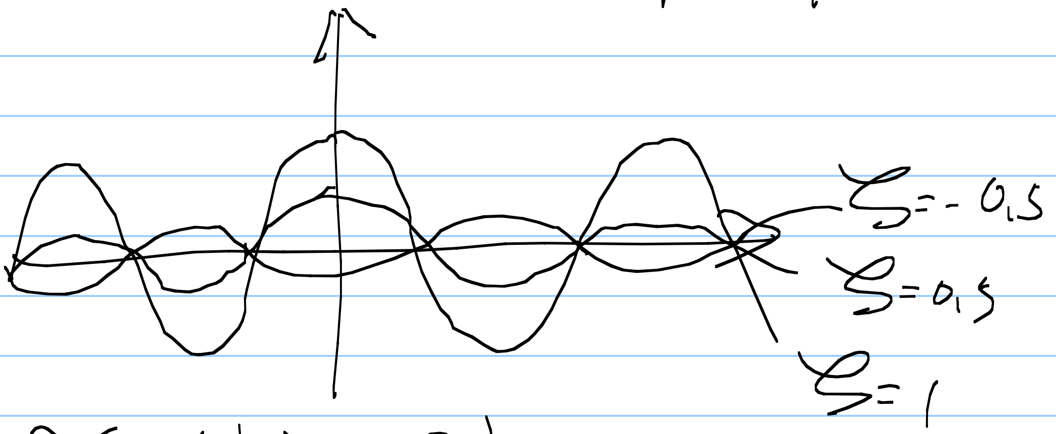
Not random

random

& ζ is uniformly distri between $[-1, 1]$

Q: What are the sample paths?

Ans:



Q: $P(X(\frac{1}{4}) = 0)$

Ans: $= 1.$

Q: $P(|X(0) - X(\frac{1}{2})| < 0.5)$

Ans: $P(|\zeta \cos(2\pi \times 0) - \zeta \cos(2\pi \times \frac{1}{2})| < 0.5)$

$= P(|2\zeta| < 0.5)$

$= P(-0.25 < \zeta < 0.25)$

$= \int_{-0.25}^{0.25} \frac{1}{2} \, d\theta = \frac{1}{4}$ #

Method 2: Sometimes it is impossible to specify the weights for individual sample paths. For example the trend of Dow-Jones in the future.

Instead, Specify the joint pdf / pmf

for any k sample R.V.s.

Namely for any $t_1, t_2, \cdots, t_k$,

We specify the joint pdf/pmf

of the k-dim R.V. $(X(t_1), X(t_2), \cdots, X(t_k))$

Ex: $\boxed{\text{A white Gaussian R.P}}$

for any $t_1, t_2, \cdots, t_k$,

(say $k=3$, $t = \pi, -e, \sqrt{3}$)

$(X_{t_1}, \cdots, X_{t_k})$ $(X_\pi, X_{-e}, X_{\sqrt{3}})$

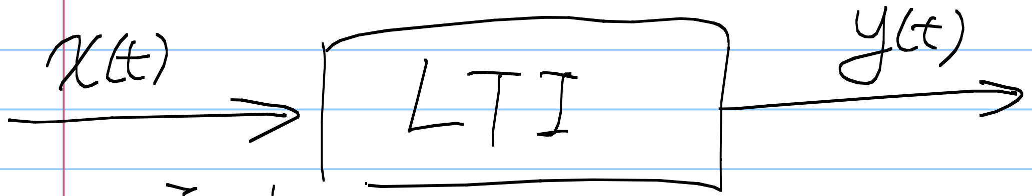
has joint pdf

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(x_{t_1})^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_{t_2})^2}{2}} \cdot \cdots\cdots \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_{t_k})^2}{2}}$$

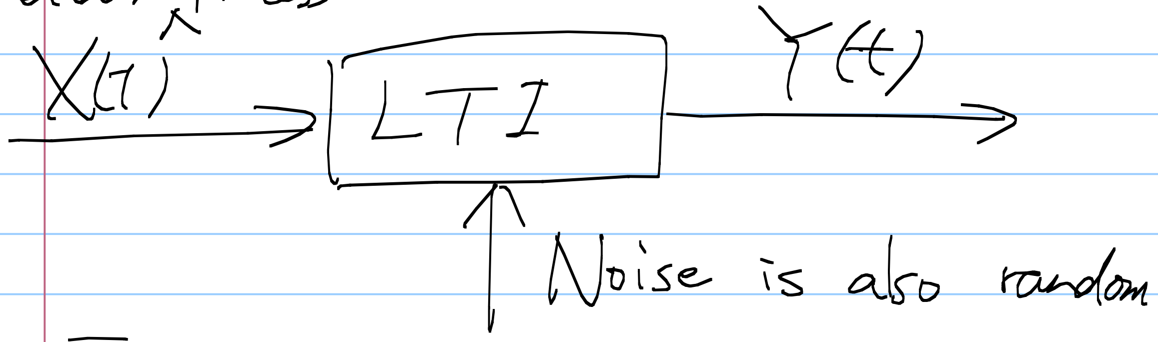
Examples 9.5, 9.6

Why studying R.P.

deterministic



Random Input Process



Interested in

P(the received audio signal $Y(t)$ becomes inaudible)=?

It is generally a very hard problem.

Recall: It is easy to find the mean & variance of a linear transform.

Therefore, most of the time we engineers focus on the mean & variance of a R.P.

☆ The $\boxed{\text{mean function}}$ of a R.P.

$m_X(t) = E(X(t))$ the mean of X(t) for a given t.

☆ The variance function

$Var[X(t)] = E((X(t) - m(t))^2)$ the variance of X(t) for a given t.

Compute it for general t_1, t_2

$E(XY)$

☆ $\underline{\text{Auto}}$ $\boxed{\text{correlation}}$ function

$$R_X(t_1, t_2) = E(X(t_1)X(t_2))$$

☆ Auto covariance function

$$C_X(t_1, t_2) = Cov(X(t_1), X(t_2))$$

How to compute the auto correlation / covariance function?

Ans: Find the joint pdf of X_{t_1}, X_{t_2} first, then treat them as X, Y & Find the expectation.

Ex: HW15 Prob 9.2.

X_n \ X_{n+k}

$\frac{1}{6}$					
	$\frac{1}{6}$				
		$\frac{1}{6}$			
			$\frac{1}{6}$		
				$\frac{1}{6}$	
					$\frac{1}{6}$

$$R_X(n, n+k) = E(X_n \cdot X_{n+k})$$

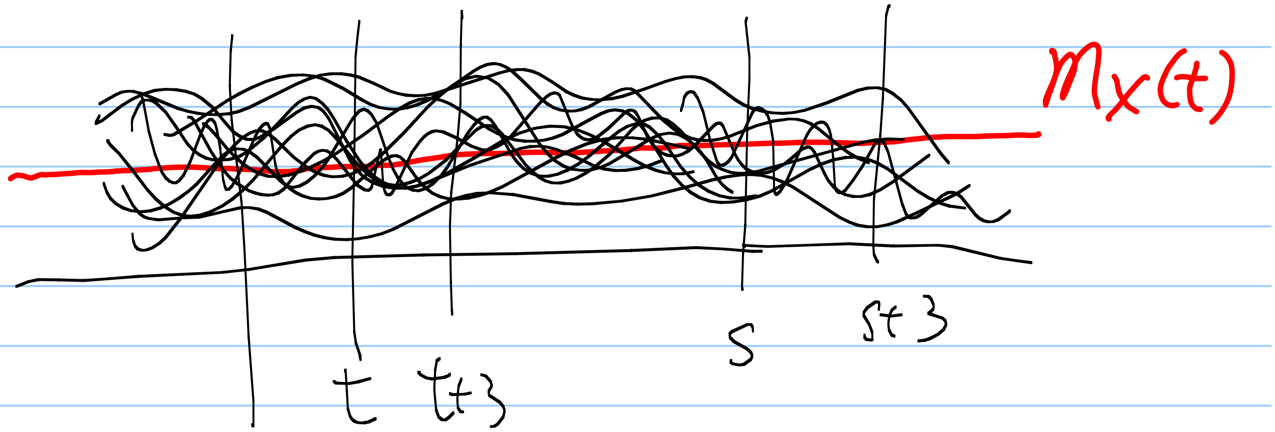
$$= 1^2 \times \frac{1}{6} + 2^2 \times \left(\frac{1}{6}\right) + \cdots + 6^2\left(\frac{1}{6}\right)$$

$$= \frac{91}{6}$$

$$C_X(n, n+k) = \operatorname{Cov}(X_n, X_{n+k})$$

$$= \frac{35}{12}$$

In many real systems the correlation & covariance depends only on the distance between t_1, t_2



& the mean function is a flat horizontal line.

We say such a R.P is

* Wide Sense Stationary (W.S.S)

& the $m_X(t) \longrightarrow \underline{\underline{m_X}}$ flat

$$R_X(t_1, t_2) \longrightarrow R_X(\Delta t)$$

I.e. $R_X(3.5)$

$= E(X_0 \cdot X_{3.5})$

or $= E(X_{100} \cdot X_{103.5})$