Note Title 4/12/2011

Comparison between $\hat{x}_{MMSE}$ and $\hat{x}_{Lin\,MMSE}$

HW13, Q5 Q6

X is equally likely on -1 and 1.

N is independent standard Gsn.

$Y = X + N$

Q: Find $\hat{x}_{MMSE}$ and $\hat{x}_{Lin\,MMSE}$

Ans: Find $\hat{x}_{MMSE}$.

Step 1: Find the conditional prob first.

$$P(X=1 \mid Y=y) = \frac{P(Y=y, X=1)}{P(Y=y, X=1) + P(Y=y, X=-1)}$$

$$= \frac{P(N=y-1, X=1)}{P(N=y-1, X=1) + P(N=y+1, X=-1)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} \times \frac{1}{2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} \times \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}} \times \frac{1}{2}}$$

$$= \frac{e^{y}}{e^{y} + e^{-y}}$$

Similarly

$$P(X=-1 \mid Y=y) = \frac{e^{-y}}{e^{y} + e^{-y}}$$

step 2: Find the conditional expectation

$$\hat{x}_{MMSE} = 1 \times \frac{e^{y}}{e^{y} + e^{-y}} + (-1) \frac{e^{-y}}{e^{y} + e^{-y}}$$

Ans Find $\hat{x}_{LIN,MMSE}$. We need to find $m_X, m_Y, \sigma_X, \sigma_Y, \rho_{XY}$.

$\hat{x}_{Lin,MMSE}$.

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$m_X = 0 \quad m_N = 0$

$m_Y = m_X + m_N = 0$.

$\sigma_X^2 = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} - 0^2$

$= 1$

$\sigma_N^2 = 1, \quad \sigma_Y^2 = \sigma_X^2 + \sigma_N^2 = 2$

$E(XY) = E(X^2 + XN) = 1$

$Cov(X,Y) = 1 - 0 = 1$

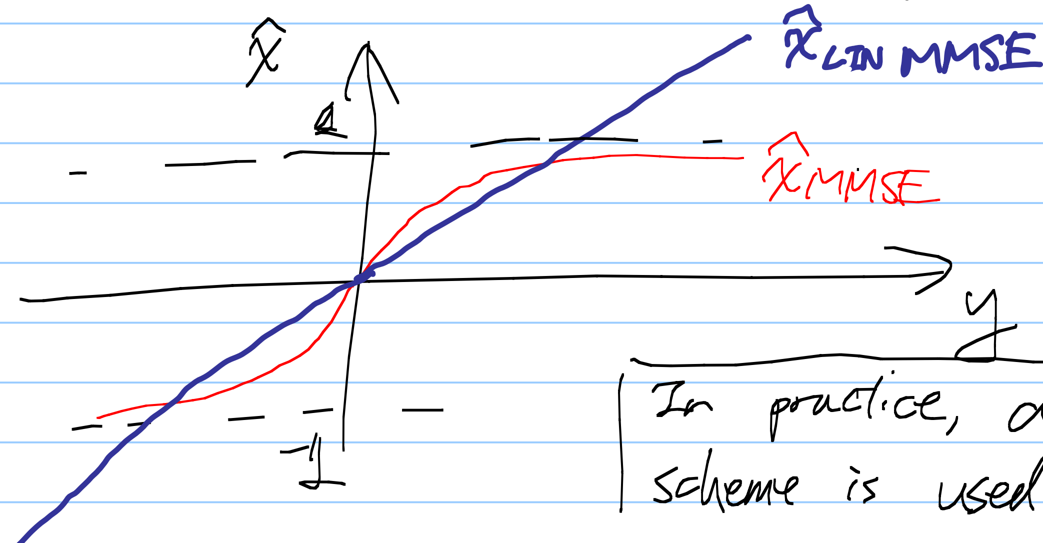
$$\rho_{XY}(x,y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{1}{\sqrt{2}}$$

By the linear MMSE formula

$$\Rightarrow \hat{x}_{LIN,MMSE} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} (y-0) + 0$$

$$= \frac{y}{2}$$

Plot $\hat{x}_{MMSE}(y)$ and $\hat{x}_{LIN\,MMSE}(y)$



In practice, a hybrid scheme is used.

$\hat{x}_{LIN\,MMSE}$ easier, but has overshoot.

$\hat{x}_{MMSE}$, always closest but hard to evaluate.

* n-dimensional R.V.

Suppose we have n different random experiments $X_1, \cdots, X_n$

How to find $P(X_1 + X_2 + \cdots + X_n \le 1)$

$E(X_1 \cdot X_2 \cdot X_3 \cdot \cdots \cdot X_n)$

Again, we treat it as a big vector $(X_1, X_2, \cdots, X_n)$

The $\boxed{\text{joint sample space}}$ is

$S = \{\text{all } n\text{-dim vectors}\}$.

the weight assignment

discrete: joint prob mass function

$$\sum_{x_1}\sum_{x_2}\sum\cdots\sum_{x_n} P_{x_1, x_2, \cdots, x_n} = 1.$$

Continuous: joint prob density function

$$\int_{x_1}\int_{x_2}\cdots\int_{x_n} f_{X_1,\cdots,X_n}(x_1, x_2, \cdots, x_n)\, dx_n\, dx_{n-1} \cdots dx_1$$

$= 1$

How to find out the prob

$P(X_1 + \cdots + X_n \leq 1)$

Counting / Integration

How to find $E(X_1 \cdot X_2 \cdot \cdots \cdot X_n)$

The weighted avg.

$$= \sum_{x_1} \sum_{\cdots} \cdots \sum_{x_n} (x_1 \cdot x_2 \cdots x_n) P_{x_1 \cdots x_n}$$

or

$$\int \int \int (x_1 \cdot x_2 \cdots x_n) f_{x_1 \cdots x_n}(x_1, \cdots, x_n) dx_1 \cdots dx_n$$

The principle is exactly the same

Ex: Prob 6.7

X, Y, Z have joint pdf

$f_{XYZ}(x, y, z) = k(x+y+z)$ for

$0 \leq x \leq 1$

$0 \leq y \leq 1$

$0 \leq z \leq 1.$

Q: Find k.

Ans: $\int_z \int_y \int_x k(x+y+z)\, dx\, dy\, dz$

$= \frac{3}{2} k = 1 \Rightarrow k = \frac{2}{3}$

Q Find the marginal pdf

$f_Y(y)$ and $f_{YZ}(y,z)$

Ans: Integrating out the variables you are not interested.

$$f_{YZ}(y,z) = \int_{x=0}^{1} \frac{2}{3}(x+y+z)\,dx$$

$$= \frac{1}{3} + \frac{2}{3}(y+z) \quad \text{if } 0 \le y \le 1,\ 0 \le z \le 1$$

$$f_Y(y) = \int_{z=0}^{1} \frac{1}{3} + \frac{2}{3}(y+z)\,dz$$

$$= \frac{2}{3}y + \frac{2}{3} \quad \text{if } 0 \le y \le 1$$

Q: Find the conditional pdf

$$f_{X|YZ}(x|y,z)$$

$$= \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)}$$

$$= \begin{cases} \dfrac{\frac{2}{3}(x+y+z)}{\frac{1}{3}+\frac{2}{3}(y+z)} & \text{if } 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

* Chain Rule for the conditional prob.

* Example: Prob 6.9. $\boxed{210}$

Show that

$$P_{X_1,X_2,X_3}(x_1,x_2,x_3)$$

$$= P_{X_1}(x_1)\cdot P_{X_2|X_1}(x_2|x_1)\cdot P_{X_3|X_1,X_2}(x_3|x_1,x_2)$$

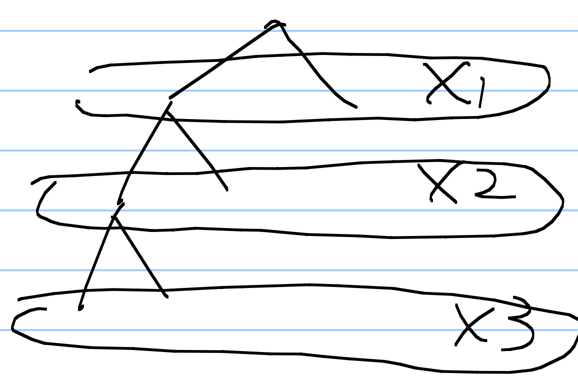
Ans: $P_{X_1,X_2,X_3}(x_1,x_2,x_3)$

$$= P(X_1=x_1, X_2=x_2, X_3=x_3)$$

$$= P(X_1=x_1)\,\frac{P(X_2=x_2, X_1=x_1)}{P(X_1=x_1)}\,\frac{P(X_3=x_3, X_2=x_2, X_1=x_1)}{P(X_2=x_2, X_1=x_1)}$$

$$= P(X_1=x_1)\,P(X_2=x_2|X_1=x_1)\,P(X_3=x_3|X_1=x_1, X_2=x_2)$$

Think about it from the tree method



Ex: X_1, X_2 are independent uniform R.Vs on $(0,2)$. Given $X_1=x_1$, $X_2=x_2$, X_3 is a

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Gsn R.V with mean $= 0$, variance $x_1^2+x_2^2$

Find the joint pdf.

Ans: $f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} \frac{1}{2}\times\frac{1}{2}\times\frac{1}{\sqrt{2\pi x_1^2+x_2^2}}\, e^{-\frac{x_3^2}{2(x_1^2+x_2^4)}} & \text{if } 0\le x_1\le 2,\ 0\le x_2\le 2 \\ 0 & \text{otherwise} \end{cases}$

* The joint cdf

$F_{X_1\cdots X_n}(x_1\cdots,x_n) = P(X_1\le x_1,\cdots,X_n\le x_n)$

* Independence:

If $X_1,\cdots,X_n$ are independent

$\Longleftrightarrow$ $P_{X_1,\cdots X_n}(x_1,\cdots,x_n)$

$= P_{X_1}(x_1)\cdot P_{X_2}(x_2)\cdots\cdots P_{X_n}(x_n)$

the joint = the product of the marginals

$f_{X_1\cdots X_n}(x_1,\cdots x_n)$

$= f_{X_1}(x_1)\cdots f_{X_n}(x_n)$

$\Longleftrightarrow$ $F_{X_1\cdots X_n}(x_1,\cdots,x_n)$ the joint cdf

$= F_{X_1}(x_1)\cdot F_{X_2}(x_2)\cdots F_{X_n}(x_n)$

$\Longleftrightarrow$ $P(a_1\le X_1\le a_2,\ b_1\le X_2\le b_2,\ c_1\le X_3\le c_2)$

$= P(a_1\le X_1\le a_2)P(b_1\le X_2\le b_2)P(c_1\le X_3\le c_2)$

* Expectation

$$E(g_1(X_1, X_2, X_3) + g_2(X_1, X_2, X_3))$$
$$= E(g_1(X_1, X_2, X_3)) + E(g_2(X_1, X_2, X_3))$$

Generally
$$E(g_1(X_1) \cdot g_2(X_2) \cdot g_3(X_3)) \neq$$
$$E(g_1(X_1)) \cdot E(g_2(X_2)) \cdot E(g_3(X_3))$$

If $X_1 \cdots X_3$ are independent

$$\Rightarrow E(g_1(X_1) \cdot g_2(X_2) \cdot g_3(X_3))$$
$$= E(g_1(X_1)) \cdot E(g_2(X_2)) \cdot E(g_3(X_3))$$

Ex: X_1 is Poisson with $\alpha = 2$
X_2 is exponential with $\lambda = 0.5$
X_3 is standard Gsn. $\qquad \because \Phi_X(\omega) = E(e^{j\omega X})$
X_1, X_2, X_3 are indep.

Find $E(X_1 X_2^2 e^{jX_3})$ $\qquad \Phi_X(1)$

$$= E(X_1) \cdot E(X_2^2) \cdot E(e^{jX_3})$$
$$= 2 \times \left(\frac{1}{(0.5)^2} + \left(\frac{1}{0.5}\right)^2\right) \cdot e^{-\frac{1}{2}} = 16 \times e^{-\frac{1}{2}}$$

* Multi-dimensional Gsn RV

$(X_1, X_2, \cdots, X_n)$ are jointly Gsn

Sample space: {all n-dimensional vectors}

Joint pdf: $f_{X_1,\cdots,X_n}(x_1,\cdots,x_n)$

$\boxed{p. 325}$

$$= \frac{1}{(\sqrt{2\pi})^n \sqrt{|K|}} e^{-\frac{1}{2}(x_1-m_1, x_2-m_2, \cdots, x_n-m_n) K^{-1} \begin{pmatrix} x_1-m_1 \\ x_2-m_2 \\ x_3-m_3 \\ \vdots \\ x_n-m_n \end{pmatrix}}$$

where K is the $\boxed{\text{Covariance Matrix}}$

$$\begin{pmatrix} Var(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) & \cdots & Cov(X_1,X_n) \\ Cov(X_2,X_1) & Var(X_2) & Cov(X_2,X_3) & & \\ \vdots & \vdots & \vdots & & \\ & & & & Var(X_n) \\ Cov(X_n,X_1) & \cdots & \cdots & & \end{pmatrix}$$

$|K|$ is the $\boxed{\text{determinant}}$ of K.

K^{-1} is the $\boxed{\text{inverse matrix}}$ s.t. $K \cdot K^{-1} = I$ the identity matrix

$(x_1-m_1, x_2-m_2, \cdots, x_n-m_n)$ is a $\boxed{\text{row vector}}$

$\begin{pmatrix} x_1-m_1 \\ x_2-m_2 \\ x_n-m_n \end{pmatrix}$ is a $\boxed{\text{column vector}}$

Properties of joint Gsn:

① The joint pdf (W.A) is determined by the means $m_1, \cdots, m_n$, variances $\sigma_1^2, \cdots, \sigma_N^2$, and all pairs of covariance $Cov(X_1, X_2)$, $Cov(X_1, X_3) \cdots$, $Cov(X_{n-1}, X_n)$

$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_n \end{pmatrix}$ is the $\boxed{\text{mean vector}}$ $K = \begin{pmatrix} Var & Cov & \cdot \\ Cov & Var & \\ \cdot & & \ddots \end{pmatrix}$ the covariance matrix.

* For general R.Vs. it is possible that X_1, X_2, X_3 have the same means variances, and covariances as Y_1, Y_2, Y_3 but they have different weight assignments.

* But for joint Gsns Z_1, Z_2, Z_3 and W_1, W_2, W_3. If (Z_1, Z_2, Z_3) have the same means, variances, and covariances as (W_1, W_2, W_3), then they have the same joint pdf

Example: X_1, X_2, X_3 are joint Gsn with means 0, variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ & covariances 0.

(X_1, X_2 are uncorrelated,
X_1, X_3 - - - - - - -
X_2, X_3 are uncorrelated)

Q: Find the joint pdf.

Ans: $K = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix}$

$|K| = \sigma_1^2 \sigma_2^2 \sigma_3^2$

$K^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & 0 \\ 0 & \frac{1}{\sigma_2^2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3^2} \end{pmatrix}$

$f_{X_1 X_2 X_3}(x_1, x_2, x_3)$

$$= \frac{1}{\sqrt{2\pi}^3 \times \sigma_1 \times \sigma_2 \times \sigma_3} e^{-\frac{1}{2}(x_1, x_2, x_3)\begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \frac{1}{\sigma_2^2} & \\ & & \frac{1}{\sigma_3^2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$$

$$= \frac{1}{\sqrt{2\pi}^3 \sigma_1 \sigma_2 \sigma_3} e^{-\frac{1}{2}(x_1, x_2, x_3)\begin{pmatrix} \frac{x_1}{\sigma_1^2} \\ \frac{x_2}{\sigma_2^2} \\ \frac{x_3}{\sigma_3^2} \end{pmatrix}}$$

$$= \frac{1}{\sqrt{2\pi}^3 \sigma_1 \sigma_2 \sigma_3} e^{-\frac{x_1^2}{2\sigma_1^2} - \frac{x_2^2}{2\sigma_2^2} - \frac{x_3^2}{2\sigma_3^2}}$$

Q: Are X_1, X_2, X_3 independent?

Ans: Yes. $f_{X_1 X_2 X_3}(x_1, x_2, x_3)$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{x_1^2}{2\sigma_1^2}}\right)\left(\frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{x_2^2}{2\sigma_2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}\,\sigma_3} e^{-\frac{x_3^2}{2\sigma_3}}\right)$$

Property ②

$\boxed{\text{Uncorrelated}}$ joint Gsns are $\boxed{\text{independent}}$.

Property ③

Independent (marginally) Gsns are joint Gsn

Property ④

* If $X_1, X_2, \cdots X_5$ are "joint Gsn" then marginally X_1 is Gsn, (X_1, X_2) is joint Gsn

* Any linear combination of a joint Gsn

$W_1 = X_1 + X_2 + X_3 + X_4 + X_5$

$W_2 = X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5$

are jointly Gsn.

* Functions of multiple R.Vs.

Ex: X_1, X_2, X_3 has joint pdf
$f_{X_1,X_2,X_3}(x_1, x_2, x_3)$
$W = \max(X_1, X_2, X_3)$

Q: How to find $f_W(w)$, the pdf of W in a step-by-step way.

Ans: Step 1: Find the cdf of W.

$$F_W(w) = P(\max(X_1, X_2, X_3) \le w)$$

$$= P(X_1 \le w, X_2 \le w, X_3 \le w)$$

$$= \int_{-\infty}^{w}\int_{-\infty}^{w}\int_{-\infty}^{w} f_{X_1X_2X_3}(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3$$

Step 2:

$$f_W(w) = \frac{d}{dw} F_W(w)$$

* Linear functions of multiple R.V.

$Z = a_1 X_1 + a_2 X_2 + a_3 X_3 + \cdots + a_n X_n$

$(X_1, X_2, \ldots, X_n)$ can be any joint distribution

Q: $m_Z = ?$

Ans: $m_Z = a_1 m_1 + a_2 m_2 + a_3 m_3 + \cdots + a_n m_n$

Q: $Var(Z) = ?$

Ans: $Var(Z) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \cdots$
$\underline{+ a_n^2 Var(X_n)}$

principal terms

$+ 2a_1 a_2 Cov(X_1, X_2) + 2a_1 a_3 Cov(X_1, X_3)$
$\cdots$
$\underline{+ 2a_{n-1} a_n Cov(X_{n-1}, X_n)}$

the cross terms.

Q: $Cov(Z, X_3) = ?$

Ans: (Think it as $Z \cdot X_3 = (a_1 X_1 + \cdots + a_n X_n) X_3$)

$= a_1 Cov(X_1, X_3) + a_2 Cov(X_2, X_3)$
$+ a_3 Var(X_3) + \cdots + a_n Cov(X_n, X_3)$

* Characteristic function of

$$Z = a_1X_1 + a_2X_2 + a_3X_3, \quad X_1, X_2, X_3 \text{ are indep}$$

Ex:

$Z = 3X$, X has characteristic function $\Phi_X(\omega)$

Find the characteristic function of Z in terms of $\Phi_X(\omega)$.

Ans: $\Phi_Z(\omega) = E(e^{j\omega Z})$

$$= E(e^{j\underline{\omega 3}X})$$

$$= \Phi_X(3\omega)$$

Ex: $Z = X + Y$. X, Y are $\boxed{\text{indep}}$ & have char. functions $\Phi_X(\omega), \Phi_Y(\omega)$

Find $\Phi_Z(\omega)$ in terms of $\Phi_X(\omega)$ $\Phi_Y(\omega)$

Ans: $\Phi_Z(\omega) = E(e^{j\omega(X+Y)})$

$$= E(e^{j\omega X} e^{j\omega Y})$$

$$= E(e^{j\omega X})E(e^{j\omega Y})$$

$$= \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

* When X, Y are indep, the char. of $Z = X + Y$ is the product of individual char.

* When $Z = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$

& $X_1, \cdots, X_n$ are indep.

then $\Phi_Z(\omega) = \Phi_{X_1}(a_1\omega) \cdot \Phi_{X_2}(a_2\omega)$

$\cdot \cdots \cdot \Phi_{X_n}(a_n\omega)$

* Let's focus on the simplest linear function: $\boxed{\text{the \underline{summation} of \underline{indep} R.V.s}}$

$Z = X + Y.$

$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$

Recall $\Phi_X(\omega) = E(e^{j\omega X})$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

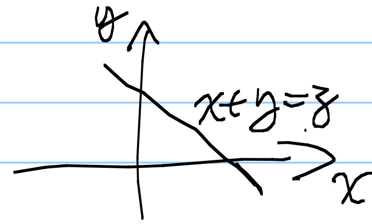
is the "freq domain" of the pdf

$f_X(x), f_Y(y) \longleftrightarrow \Phi_X(\omega), \Phi_Y(\omega)$

$\downarrow$ convolution in time $\qquad\qquad$ $\downarrow$ multication in freq

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx \longleftrightarrow \Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

summing over the $x+y=z$ line.



* Summary $Z = X + Y$ & X, Y are indep

(freq dom) $\Phi_Z(\omega) = \Phi_X(\omega)\,\Phi_Y(\omega)$ multi in freq

(time dom) $f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$ conv in time.

* Summation ~~of~~ important R.Vs. (& indep.)

① Bernoulli + Bernoulli

X, Y are indep Bernoulli with a <u>common</u> parameter p.

$Z = X + Y$. What is the distribution of Z

Ans: $\Phi_X(\omega) = (1 - p + pe^{j\omega})$ table lookup

$\Phi_Y(\omega) = (1 - p + pe^{j\omega})$

$\Phi_Z(\omega) = (1 - p + pe^{j\omega})^2$

↳ by table look up.

Z is binomial with $n = 2$, p.

(The number of heads after flipping a coin twice.)

② binomial + binomial

X_1: binomial with para n_1, p

X_2: - - - - - - ——— - n_2, p.

$Z = X_1 + X_2$ The distribution of Z.

Ans: $\Phi_{X_1}(\omega) = (1-p+pe^{j\omega})^{n_1}$

$\Phi_{X_2}(\omega) = (1-p+pe^{j\omega})^{n_2}$

$\Phi_Z(\omega) = (1-p+pe^{j\omega})^{n_1+n_2}$

$\Rightarrow Z$ is binomial with $n = n_1 + n_2$, p
(Coin flipping)

③ Poisson + Poisson

X: Poisson with para α_1

Y: Poisson - - ——— α_2

$Z = X + Y$, distribution of Z = ?

Ans: $\Phi_X(\omega) = e^{\alpha_1 (e^{j\omega} - 1)}$

$\Phi_Y(\omega) = e^{\alpha_2 (e^{j\omega} - 1)}$

$\Phi_Z(\omega) = e^{\alpha_1 (e^{j\omega} - 1)} \cdot e^{\alpha_2 (e^{j\omega} - 1)}$

$= e^{(\alpha_1 + \alpha_2)(e^{j\omega} - 1)}$

Z: Poisson with para $\alpha_1 + \alpha_2$

The number of arrivals with avg arrival rate $\alpha_1 + \alpha_2$

④ Gsn + Gsn $\Rightarrow$ Gsn [Any linear op. on Gsn gives Gsn