Note Title 3/25/2011

* Revisit Independence (four equivalent def'n)

X & Y are indep.

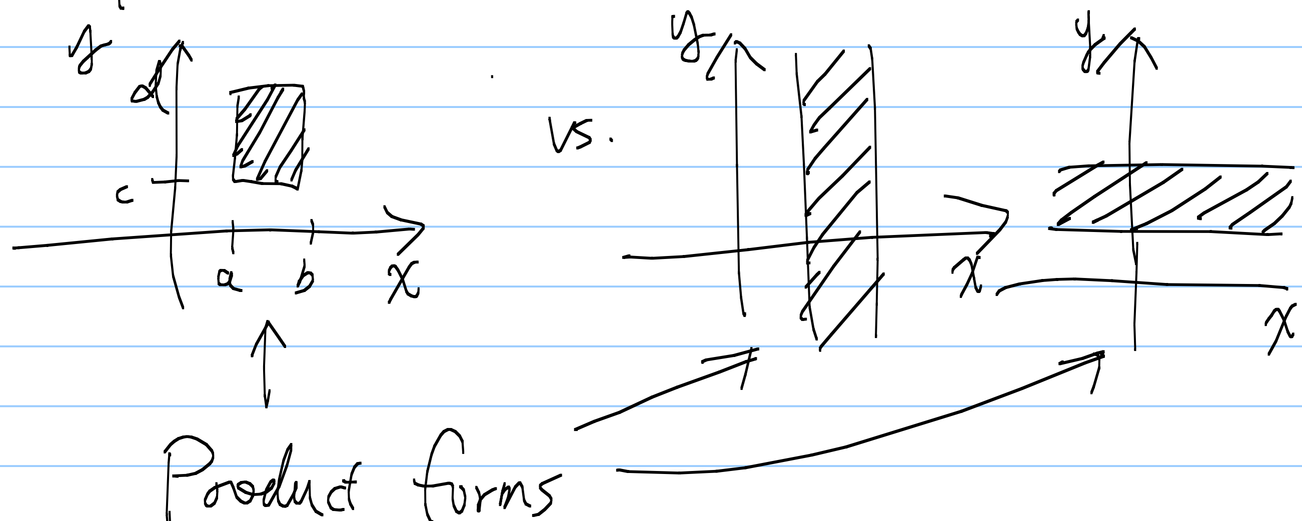
$\Leftrightarrow$ $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

or $p_{k,h} = p_k \cdot p_h$

$\Leftrightarrow$ $\boxed{F_{XY}(x,y) = F_X(x) \cdot F_Y(y)}$

$\Leftrightarrow$ $P(a \le X \le b,\ c \le Y \le d)$

$= P(a \le X \le b) \cdot P(c \le Y \le d)$



* To check independence

$F_{XY}(x,y) \xrightarrow{\text{Step 1}}$ Compute $F_X(x)$ & $F_Y(y)$

Step 2: Check whether $F_{XY}(x,y) \stackrel{?}{=} F_X(x) \cdot F_Y(y)$

Example: Q: Are R & Θ in the previous example independent?

Ans:

$$F_{R,\Theta}(r,\theta)=\begin{cases} 0 & \text{if } r<0 \text{ or } \theta<0 \\ r^2\cdot\frac{\theta}{2\pi} & \text{if } 0\le r<1 \text{ and } 0\le\theta<2\pi \\ r^2 & \text{if } 0\le r<1,\ 2\pi\le\theta \\ \frac{\theta}{2\pi} & \text{if } 1\le r,\ 0\le\theta<2\pi \\ 1 & \text{if } 1\le r \text{ and } 2\pi\le\theta \end{cases}$$

$$F_R(r)=\begin{cases} 0 & \text{if } r<0 \\ r^2 & \text{if } 0\le r<1 \\ 1 & \text{if } 1\le r \end{cases}$$

$$F_\Theta(\theta)=\begin{cases} 0 & \text{if } \theta<0 \\ \frac{\theta}{2\pi} & \text{if } 0\le\theta<2\pi \\ 1 & \text{if } 2\pi\le\theta \end{cases}$$

Since $F_{R,\Theta}(r,\theta)=F_R(r)\cdot F_\Theta(\theta)$

$\Rightarrow$ Yes. They are indep.

$\boxed{\text{Don't forget that we can also check } f_{R,\Theta}(r,\theta)\overset{?}{=}f_R(r)\cdot f_\Theta(\theta)}$

* Revisit Expectation

conti 2-dim R.V

$$E(X^2Y+e^{X+Y})$$
$$=\iint (x^2y^2+e^{x+y})f_{XY}(x,y)dy$$

discrete

$$E(X^2Y+e^{X+Y})$$
$$=\sum_k\sum_h (k^2h+e^{k+h})p_{kh}$$

* Properties of expectations.

① Expectation of a constant is the constant itself

② $E(g_1(X,Y) + g_2(X,Y))$

$= E(g_1(X,Y)) + E(g_2(X,Y))$

Ex: $E(X^2Y + e^{XY})$

$= E(X^2Y) + E(e^{XY})$

Why? The weighted average formula

③ In general $E(XY) \neq E(X)E(Y)$

Ex:

$X \backslash Y$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	$\frac{1}{3}$	0

Q: $E(X)$, $E(Y)$, $E(XY)$

Ans: $E(XY) = 0 \quad E(X) = \frac{1}{3} \quad E(Y) = \frac{1}{3}$

$E(XY) \neq E(X) \cdot E(Y)$

Similarly $E(X^2 e^Y) \neq E(X^2)\cdot E(e^Y)$

④ However, if X & Y are indep and the product can be expressed as $\underbrace{g_1(X)}_{\text{only } X}\underbrace{g_2(Y)}_{\text{only } Y}$

then $E(g_1(X)\, g_2(Y)) = E(g_1(X))\cdot E(g_2(Y))$

pf: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_1(x) g_2(y))\cdot f_{XY}(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_1(x) g_2(y)) \boxed{f_X(x) f_Y(y)}\, dx\,dy$$

↳ independence

$$= \int_{-\infty}^{\infty} g_2(y) f_Y(y) \left(\int_{-\infty}^{\infty} g_1(x) f_X(x)\, dx\right) dy$$

$$= \left(\int_{-\infty}^{\infty} g_1(x) f_X(x)\, dx\right)\left(\int_{-\infty}^{\infty} g_2(y) f_Y(y)\, dy\right)$$

Ex: X is standard Gsn, Y is exponential with λ, X & Y are independent. Find $E(X^2 Y)$

Ans $E(X^2Y) = E(X^2)E(Y)$ only when X, Y are indep

$= 1 \times \frac{1}{\lambda}$

Ex: We can not express

$E(X^2) = E(X\cdot X) \underset{\text{wrong}}{=} E(X)\cdot E(X)$. Since X is "dependent" of itself

Note Title 3/28/2011

Note: $E(X^2 e^X)$

$\neq (E(X)) \cdot (E(X)) \cdot E(e^X)$

$\because$ X is NOT independent of X

* Revisit Conditional Expectation

Ex:

X \ Y	0	1	2	
0	$\frac{1}{2} \times \frac{1}{4}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{2} \times \frac{1}{9}$	$\frac{1}{2} \times \frac{4}{9}$	$\frac{1}{2} \times \frac{4}{9}$	$\frac{1}{2}$

Q: Find $E(e^X Y \mid X = x)$

which is a function ~~of~~ x, denote it by $G(x)$.

Ans: when $x=0$. $E(e^X Y \mid X=0)$

$= E(e^0 Y \mid X=0)$

$= e^0 \times 2 \times (\frac{1}{2}) = 1$

(Given $X=0$ Y is a binomial

$n=2$ $p=0.5$

when $x=1$ $E(e^X Y \mid X=1)$

$$= E(e^1 Y \mid X=1)$$

$$= e^1 \times 2 \times \left(\frac{2}{3}\right) = \frac{4}{3} e^1$$

$$\Rightarrow G(x) = \begin{cases} 1 & \text{if } x=0 \\ \frac{4e}{3} & \text{if } x=1 \\ 0 & \text{otherwise.} \end{cases}$$

Q: $E(G(X)) = ?$

Ans $= \frac{1}{2} \times 1 + \frac{4e}{3} \times \frac{1}{2} = \frac{1}{2} + \frac{2e}{3}$ ※

Q: $E(e^X Y) = ?$

$$= \frac{1}{8} \times 0 + \frac{1}{4} \times 1 + \frac{1}{8} \times 2$$
$$+ \frac{1}{18} \times 0 + \frac{4}{18} \times e^1 + \frac{4}{18} \times e^1 \times 2$$

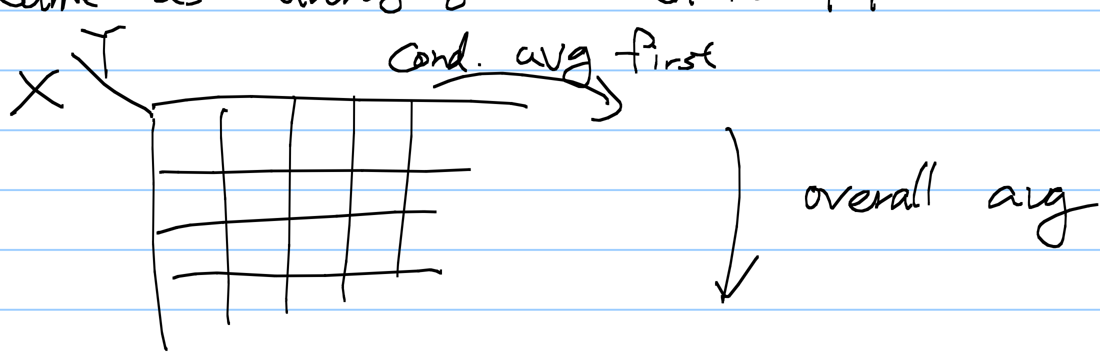
$$= \frac{1}{2} + \frac{2e}{3}$$

It is not a coincidence

☆ For any $g(x, y)$. Let $G(x) = E(g(X, Y) \mid X = x)$

then $E(G(X)) = E(g(X, Y))$

Why: averaging the averages ~~of~~ the subgroups is the same as averaging the entire population.



Since $G(x) = E(g(X,Y) \mid X = x)$

$\Rightarrow G(X) = E(g(X,Y) \mid X)$

It is commonly written in the following form

$$E(g(X,Y)) = E(E(g(X,Y) \mid X))$$

* The above relationship is usually used as a tool "computing the expectation from conditional expectation"

Ex: Similar to Prob 5.86

X : Bernoulli with $p = 1/3$

Given $X=0$, Y is exponential with $\lambda = 3$, Given $X=1$, Y is Poisson with $\alpha = 2$

Find $E(Y)$, Find $Var(Y)$

Ans: let $G(x) = E(Y \mid X = x)$

then $E(Y) = E(G(X))$

$G(0) = E(Y \mid X=0) = \frac{1}{\lambda} = \frac{1}{3}$

$G(1) = E(Y \mid X=1) = \alpha = 2$

$E(G(X)) = \frac{1}{3} \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{8}{9}$ #

Q: Find $Var(Y)$.

Ans: $Var(Y)$ is not an expectation, we do not have $E(Var(Y|X)) = Var(Y)$ ⟵ wrong

However, we have

$$Var(Y) = E(Y^2) - (E(Y))^2$$

Find $E(Y^2)$ first. Let $g(x,y) = y^2$

let $G(x) = E(Y^2 | X = x)$

$$G(0) = \frac{1}{\lambda} + m^2$$

$$= \frac{1}{3} + \left(\frac{1}{3}\right)^2 = \frac{4}{9}$$

$$G(1) = \alpha + m^2 = 2 + 2^2 = 6$$

$$E(Y^2) = E(G(X))$$

$$= \frac{4}{9} \times \frac{2}{3} + 6 \times \frac{1}{3} = \frac{62}{27}$$

$$Var(Y) = E(Y^2) - m^2$$

$$= \frac{62}{27} - \left(\frac{8}{9}\right)^2 \quad \#$$

* Important expectations $E(g(X,Y))$

① $E(X)$, $E(Y)$ (marginal) expectation

② $Var(X)$, $Var(Y)$

③ $E(X^j \cdot Y^k)$ (ex $E(X^1 Y^2)$)

the (j,k)-th joint moment of X and Y.

☆ ④ $E(XY)$ the $\boxed{\text{correlation}}$ of X and Y

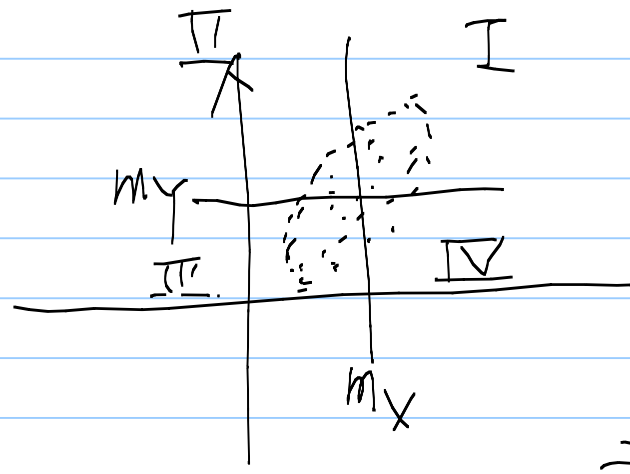
⑤ $E((X-m_X)^j \cdot (Y-m_Y)^k)$

the $(j;k)$-th central moment of X and Y

⑥ $E((X-m_X)(Y-m_Y))$: the $\boxed{\text{covariance}}$ of X and Y

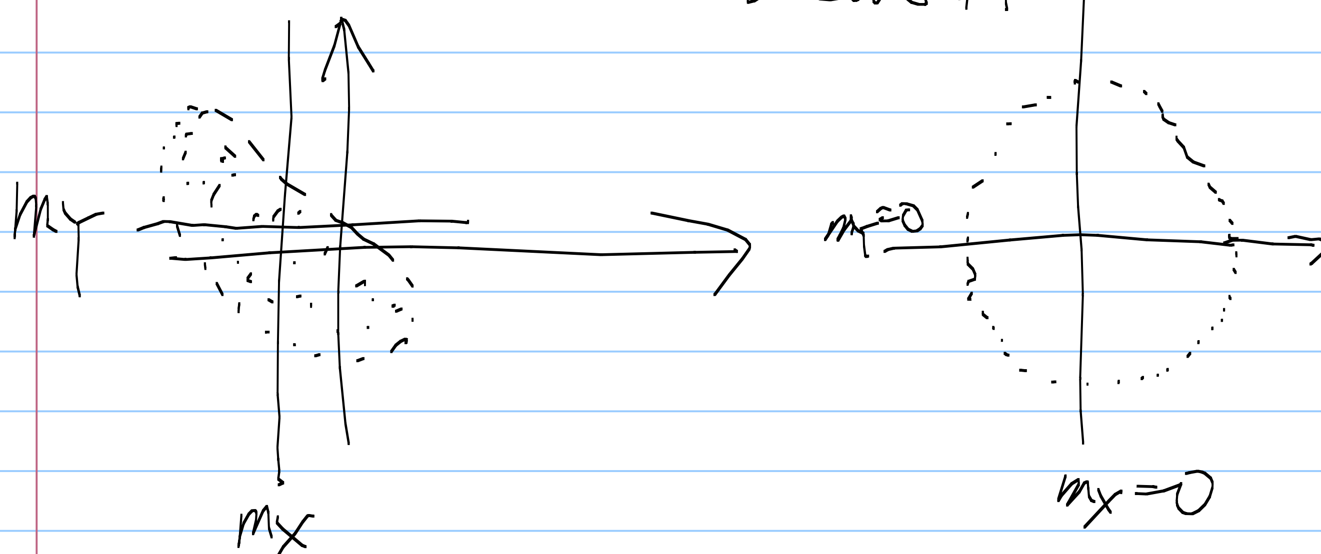
$$Cov(X,Y) \triangleq E((X-m_X)(Y-m_Y))$$

The physical meaning of $\boxed{\text{covariance}}$



the I, III quadrants outweight the II, IV quadrant

$\Rightarrow Cov(X, Y) > 0$



the II, IV quadrants outweight I, III,

$\Rightarrow Cov(X, Y) < 0$

I, III = II, IV

$\Rightarrow Cov(X, Y) = 0$

☆ Covariance $Cov(X, Y) > 0 \Rightarrow$ positively correlated

$Cov(X, Y) < 0 \Rightarrow$ negatively correlated

$Cov(X, Y) = 0 \Rightarrow$ uncorrelated

☆ Correlation $E(XY) = 0 \Rightarrow$ <u>orthogonal</u>

* An alternative formula for Cov(X, Y)

$$\boxed{Cov(X,Y) = E((X-m_X)\cdot(Y-m_Y))}$$

$$= E(XY - m_X Y - m_Y X + m_X m_Y)$$

$$= E(XY) - E(m_X Y) - E(m_Y X) + m_X m_Y$$

$$= E(XY) - m_X m_Y - m_X m_Y + m_X m_Y$$

$$\boxed{= E(XY) - m_X m_Y}$$

Example: HW11 Q3 Prob 5.65

$$f_{XY}(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Q $E(XY) = ?$

A: $\int_0^1 \int_0^1 xy(x+y)\,dx\,dy$

$$= \frac{1}{3}$$

Q: m_X, m_Y = ?

A: $E(X) = \int_0^1 \int_0^1 x(x+y)\,dx\,dy = \frac{7}{12}$

$E(Y) = \int_0^1 \int_0^1 y(x+y)\,dx\,dy = \frac{7}{12}$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= \frac{1}{3} - \left(\frac{7}{12}\right)^2$$
$$= \frac{48}{144} - \frac{49}{144} = \frac{-1}{144}$$

Q: Are X and Y orthogonal?

Ans: No. since $E(XY) \neq 0$

Q: Are X and Y correlated?

Ans: Yes. $Cov(X,Y) < 0 \Rightarrow$ negatively correlated.

Q: Are X and Y independent?

Ans: No. $\because$ $f_{XY}(x,y) = \begin{cases} (x+y) & 0 \le X \le 1,\ 0 \le Y \le 1 \\ 0 & \end{cases}$

is not a product of $f_X(x), f_Y(y)$.

Independence vs uncorrelated

* If X & Y are independent

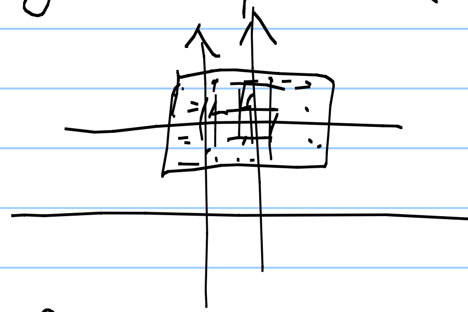
$\Rightarrow$ they are not correlated

$\because$ $Cov(X, Y) = E(XY) - E(X)E(Y)$

$\downarrow$ by independence

$= E(X)E(Y) - E(X)E(Y) = 0.$

Intuitively independence means the



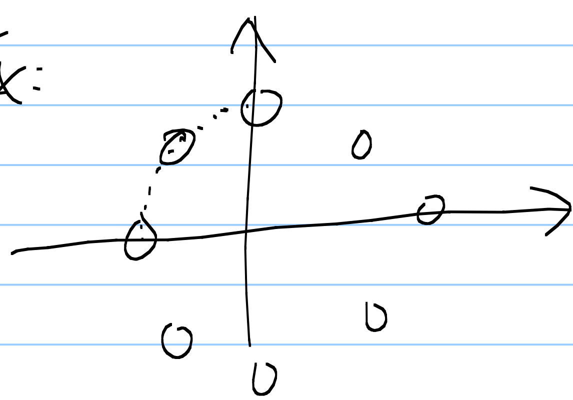
W.A is very well-behaved on a rectangular

All four quadrants (I, III) cancels each other (II, IV)

* If X & Y are not correlated

$\Rightarrow$ then they may or may not be independent

Ex:



Prob. 5.12

$Cov(X, Y) = 0$

$\because$ the quadrants cancel each other

but they are not indep. Knowing $X = 0$, or 1 changes the distribution $P(Y \mid X = x)$

Note Title 3/30/2011

Recall $\cancel{X}$ Correlation $E(XY) \xrightarrow{?}$ Orthogonal   Covariance $\xrightarrow{?}$ uncorrelated $\Uparrow$ independence

$E((X-m_X)(Y-m_Y))$
$= E(XY) - m_X m_Y$

☆ Correlation coeff

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$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

Sometimes we use $\rho(X,Y)$ as well

☆ Properties of ρ

1. $-1 \le \rho \le 1$.

2. $\rho > 0$ positively correlated
   $\rho < 0$ negatively correlated
   $\rho = 0$ uncorrelated

3. If X & Y are indep,
   then $\rho = 0$ ($\because Cov(X,Y) = 0$)
   (Note: $\rho = 0$ does not mean X & Y are indep.

Ex: X Benoulli w. p.

$$Y = 3X + 2$$

Q: $\rho(X,Y) = ?$

Ans: $Var(X) = p(1-p)$

$Var(Y) = 3^2\, Var(X) = 9p(1-p)$

$$E(XY) = E(X(3X+2))$$
$$= 3E(X^2) + 2E(X)$$
$$= 3(p(1-p) + p^2) + 2p$$
$$= 5p$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= 5p - p(3p+2)$$
$$= 3p - 3p^2$$

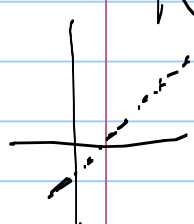
$$\rho = \frac{3p-3p^2}{\sqrt{p(1-p) \times 9p(1-p)}} = 1$$

※

4. If $\rho = 1$ then we say

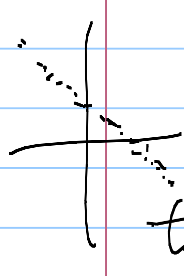
X and Y are related linearly (and positively)

Namely $Y = aX + b$ for some $a > 0$



If $\rho = -1$, then we say

X & Y are related linearly (& negatively)



$Y = -aX + b$ for some $a > 0$

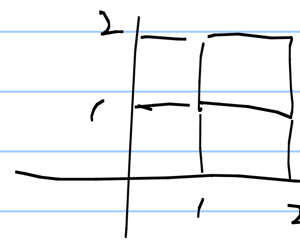
the closer ρ to 1, the more linearity X & Y have!

Revisit

* functions of 2-dim R.V.

Ex: X, Y are uniformly distributed over $(1,2)\times(1,2)$

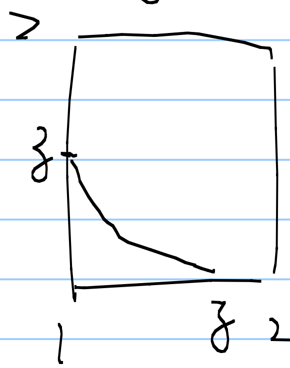
Q: $Z = XY$. find the cdf of Z.

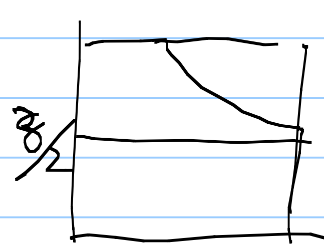


Ans: $F_Z(z) = P(Z \le z)$

$= P(X\cdot Y \le z)$

$$= \begin{cases} 0 & \text{if } z < 1 \\ \int_1^{z}\int_1^{z/y} 1\times dx\,dy = z\ln z - z + 1 & \text{if } 1 \le z < 2 \\ \int_1^{z/2}\int_1^{2} 1\times dx\,dy + \int_{z/2}^{2}\int_1^{z/y} 1\times dx\,dy = z(2\ln 2) - z\ln z + z - 3 & \text{if } 2 \le z < 4 \\ 1 & \text{if } 4 \le z \end{cases}$$





Note Title 3/30/2011

* The most important function is the linear function

$$Z = aX + bY$$

$$W = cX + eY$$

We discuss the relationship between m_Z, m_W & $Var(Z)$, $Var(W)$, $Cov(Z,W)$

① $m_Z = a m_X + b m_Y$

$m_W = c m_X + e m_Y$

② $Var(Z) = a^2 Var(X) + 2ab\, Cov(X,Y) + b^2 Var(Y)$

$\because E(Z^2) = E((aX+bY)^2) = E(a^2X^2 + 2ab\,XY + b^2Y^2)$

This gives a hint of the above formula (esp. when all means $= 0$)

Similarly, $Var(W) = c^2 Var(X) + 2ce\, Cov(X,Y) + e^2 Var(Y)$

Q: $Cov(Z,W) = ?$

Ans: $Cov(Z,W) = ac\, Var(X) + (ae+bc)\, Cov(X,Y) + be\, Var(Y)$.

pf: Because

$E(Z \cdot W) = E((aX+bY)(cX+eY))$

$= ac\,E(X^2) + (ae+bc)\,E(XY) + be\,E(Y^2)$

If X and Y are also indep.

$\Rightarrow Var(Z) = a^2 Var(X) + b^2 Var(Y)$

$\because Cov(X, Y) = 0$

HW12Q6

X & Y are indep & with means & variances m_X, σ_X^2, m_Y, σ_Y^2

Q: Find out the correlation between X & Y

Ans: $E(XY) = Cov(X, Y) + m_X m_Y$

$= 0 + m_X m_Y$ # $\because$ indep.

Q: $Z = X + Y$. Find m_Z & $E(Z^2)$

Ans: $E(Z) = E(X+Y) = E(X) + E(Y) = m_X + m_Y$

$E(Z^2) = E(X^2 + 2XY + Y^2)$

$= (\sigma_X^2 + m_X^2) + 2(m_X m_Y) + (\sigma_Y^2 + m_Y^2)$

$= \sigma_X^2 + \sigma_Y^2 + (m_X + m_Y)^2$

Q: $Var(Z)$

Ans: $= E(Z^2) - (E(Z))^2 = \sigma_X^2 + \sigma_Y^2$ #

* 2-dim Joint Gsn R.V. (X, Y)

S_{XY} : {all real 2-dim vectors}.

five input parameters.

m_X, m_Y, σ_X, σ_Y, $\underline{\rho_{XY}}$ (or just ρ)

↳ the correlation coeff

$$f_{XY}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y (\sqrt{1-\rho^2})} \times e^{-\frac{\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2}{2(1-\rho^2)}}$$

See p.229 for illustration

* Example : Prob 5.110

$$f_{XY}(x,y) = \frac{1}{2\pi \times c} e^{-2x^2 - y^2/2}$$ is a joint Gsn.

Find c, σ_X, σ_Y, and $\rho_{X,Y}$, $Cov(X,Y)$

Solved by inspection

Ans: $\sigma_X \sigma_Y \sqrt{1-\rho^2} = c$ — by inspecting the constant coeff

$$2\rho\left(\frac{1}{\sigma_X}\right)\left(\frac{1}{\sigma_Y}\right) = 0 \Rightarrow \boxed{\rho = 0}$$

by inspecting the x·y term

$$\frac{1}{2(1-\rho^2)} \times \frac{1}{\sigma_X^2} = 2 \Rightarrow \boxed{\sigma_X^2 = \frac{1}{4}}$$

$\sigma_X = \frac{1}{2}$

by inspection of the x^2 term

$$\frac{1}{2(1-\rho^2)} \times \frac{1}{\sigma_Y^2} = \frac{1}{2} \Rightarrow \boxed{\sigma_Y^2 = 1} \quad \sigma_Y = 1$$

by inspection of the y^2 term

$$\boxed{c = \frac{1}{2} \times 1 \times \sqrt{1-0^2} = \frac{1}{2}}$$

$$\boxed{Cov(X,Y) = \rho \times \sigma_X \times \sigma_Y = 0}$$

HW12Q10 Prob 5.111

$$f_{XY}(x,y) = \frac{1}{2\pi \cdot c} e^{-\frac{1}{2}(x^2+4y^2-3xy+3y-2x+1)}$$

Find m_X, m_Y, σ_X^2, σ_Y^2, ρ, $Cov(X,Y)$

Ans: We first express it as

$$\frac{1}{2\pi c} e^{-\frac{1}{2}\left(1 \times (x-a)^2 + (-3)(x-a)(y-b) + 4(y-b)^2\right)}$$

& find a, b, by inspection