

Ex = X is uniformly distributed on the interval $(0, 2)$, and given $X=x$. Y is

exponential with $\lambda = x$

Q: find the joint pdf of X, Y .

Ans: Joint pdf = marginal \times conditional

$\Rightarrow f_{XY}(x, y) = \int \underbrace{\frac{1}{2}}_{\text{marginal}} \cdot \underbrace{x e^{-xy}}_{\text{conditional}} \quad \begin{matrix} \text{if } 0 < x < 2 \\ 0 < y \end{matrix}$

0 otherwise

* Independence: the marginals \equiv conditional

$$\Rightarrow f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

Ex: X is uniform on $(0, 2)$

Y is exponential with $\lambda = 2$

X, Y are indep. Q: $f_{XY}(x, y) = ?$

Ans: $f_{XY}(x, y) = \int \frac{1}{2} 2x e^{-2y} \quad \begin{matrix} \text{if } 0 < x < 2 \\ 0 < y \end{matrix}$

0 otherwise

Summary

152

Note Title

* Continuous 2-dim R.V.s.

3/21/2011

$$S_{XY} = \{ \text{all real vectors} \}$$

W.A. Joint pdf $f_{XY}(x, y)$

the prob is the volume above the area of interest.

* Joint $\xrightarrow{\text{integration}}$ marginal

* Marginal • conditional \rightarrow joint

$$P(X=k) \cdot P(Y=h|X=k) = P(X=k, Y=h)$$

$$f_X(x) \cdot f_{Y|X}(y|x) = f_{XY}(x, y)$$

* Independence

$$\underbrace{f_{Y|X}(y|x)}_{\text{conditional}} = \underbrace{f_Y(y)}_{\text{marginal}}$$

$$\Rightarrow f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

* Expectation for 2-dim conti R.V.

$$E(g(X, Y)) \quad \left[\text{say } E(X^2 \sqrt{Y} + 3Y) \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) dy$$

HW 11 Q10 Prob 5.58

X is standard Gsn

Y is uniform on [0, 3]

X & Y are indep

Q: Find $f_{XY}(x, y)$

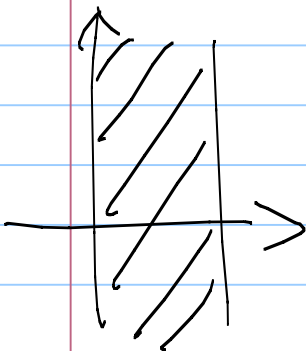
Q: $E(X^2 e^Y) = ?$

Ans: $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

$$= \int_{-\infty < x < \infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{3} \quad \text{if } 0 \leq y \leq 3$$

0

otherwise



$$Q: E(X^2 e^Y)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=0}^3 x^2 e^y \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{3} dy dx$$

$$= \frac{1}{3} \int_{x=-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} [e^y]_0^3 dx$$

$$= \frac{e^3 - 1}{3} \int_{x=-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{e^3 - 1}{3} E(X^2) = \frac{e^3 - 1}{3} (\text{Var}(X) + (E(X))^2)$$

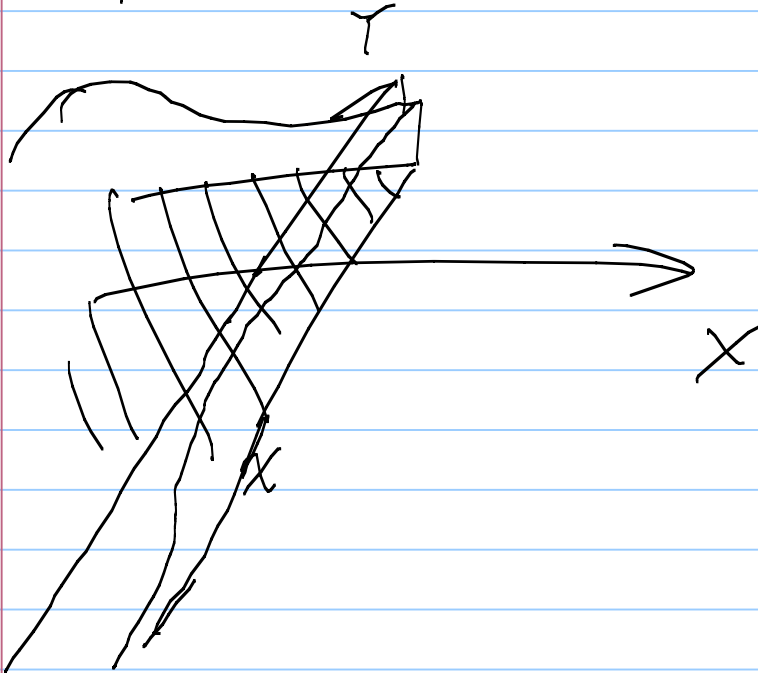
$$= \frac{e^3 - 1}{3} \times 1 \quad \#$$

* Joint cdf: a unifying way to describe 2-dim discrete/conti R.Vs.

Joint cdf

$F_{XY}(x, y)$ is a function of two para. x & y .

$$= P(X \leq x, Y \leq y)$$



Ex: X and Y have a joint pmf

		Y	
	X	1	2
0		$\frac{1}{4}$	$\frac{1}{3}$
1		$\frac{1}{3}$	$\frac{1}{2}$

Find the joint cdf $F_{XY}(x, y)$

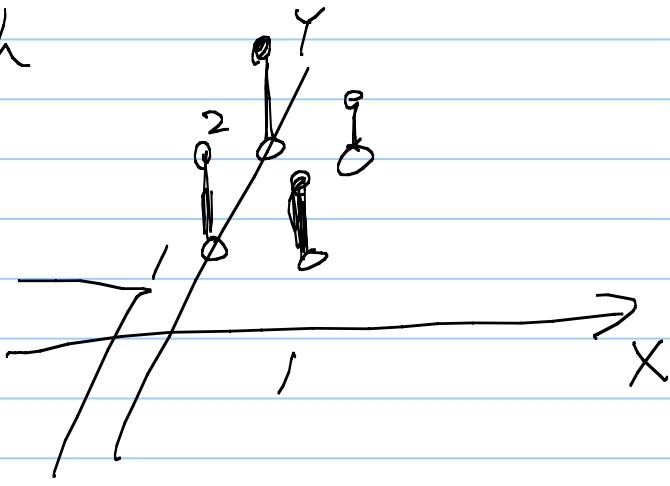
Ans: $F_{XY}(x, y) = P(X \leq x, Y \leq y)$

$$= \int \begin{matrix} 0 \\ 0 \end{matrix}$$

$$P_{0,1} = \frac{1}{4}$$

if $x < 0$
 if $y < 1$
 if $0 \leq x < 1$
 $1 \leq y < 2$

Pk.h



$$P_{0,1} + P_{1,1} = \frac{7}{12}$$

if $1 \leq x$
 $1 \leq y < 2$

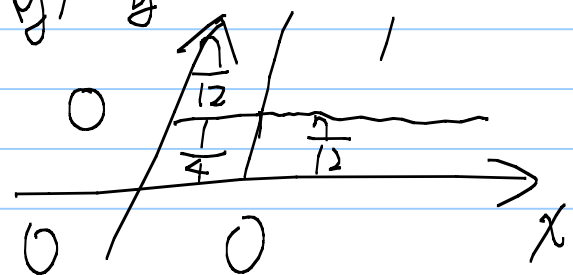
$$P_{0,1} + P_{0,2} = \frac{7}{12}$$

if $0 \leq x < 1$
 $2 \leq y$

$$P_{0,1} + P_{0,2} + P_{1,1} + P_{1,2} = 1$$

if $1 \leq x$
 $2 \leq y$

$F_{X,Y}(x,y)$



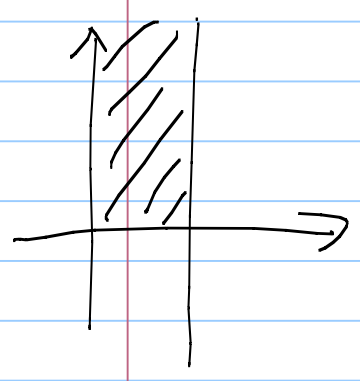
Note Title

Ex: X is uniform on $(0, 2)$

Y is exponential with $\lambda = 2$

X, Y are indep. Q: Find $F_{XY}(x, y)$

Ans: $f_{XY}(x, y) = \begin{cases} \frac{1}{2} 2x e^{-2y} & \text{if } 0 < x < 2 \\ & 0 < y \end{cases}$
 0 otherwise



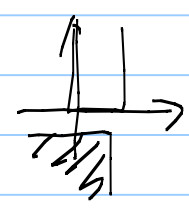
$F_{XY}(x, y) = P(X \leq x, Y \leq y)$

$= \int$

0
 0



Case 1
if $x < 0$



Case 2
if $y < 0$



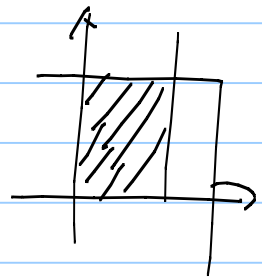
Case 3
if $0 < x < 2$
 $0 < y$

$$= \int_0^x \int_0^y \frac{1}{2} 2s e^{-2t} dt ds$$

$$= \int_0^x \frac{1}{2} (1 - e^{-2y}) ds$$

$$= \frac{x}{2} (1 - e^{-2y})$$

Case 4



if $z \leq x$
 $0 < y$

$$\int_0^z \int_0^y \frac{1}{2} z e^{-2t} dt ds = (1 - e^{-2y})$$

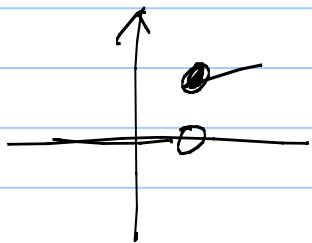
* Properties of the joint cdf

- ① $F_{XY}(x, y) \geq 0$
- ② $F_{XY}(x, y) \leq 1$
- ③ $F_{XY}(x, -\infty) = 0, F_{XY}(\infty, \infty) = 1.$

$$F_{YY}(-\infty, y) = 0$$

- ④ $F_{XY}(x_1, y_1) \leq F_{XY}(x_2, y_2)$
 if $x_1 \leq x_2$ and $y_1 \leq y_2$

- ⑤ $F_{XY}(x, y)$ is continuous from the north & east.



$$* F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

159

Note Title

3/25/2011

* Use $F_{XY}(x, y)$ to compute the prob.

The question will be of the form

Q: Given a $F_{XY}(x, y) = \begin{cases} 0 & \text{if } x < 0 \\ & \text{or } y < 0 \\ \frac{x}{2}(1 - e^{-2y}) & \text{if } 0 \leq x < 2 \\ & 0 \leq y \\ (1 - e^{-2y}) & \text{if } 2 \leq x \\ & 0 \leq y \end{cases}$

Q Find $P(X \leq 3, Y \leq 2)$

A: $= F_{XY}(3, 2) = 1 - e^{-4}$

Q: Find $P(X \leq 3)$

A: $= F_{XY}(3, \infty) = 1 - e^{-2 \cdot \infty} = 1$

Q: Find $P(Y \leq 5)$

A: $= F_{XY}(\infty, 5) = 1 - e^{-2 \cdot 5} = 1 - e^{-10}$

Q: Find $P(X < 2, Y < 5)$

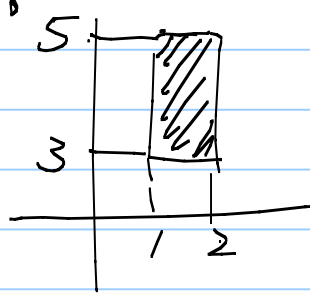
A: $= F_{XY}(2^-, 5^-) = \frac{2}{2}(1 - e^{-2 \cdot 5}) = 1 - e^{-10}$

it only matters when either $x=2$

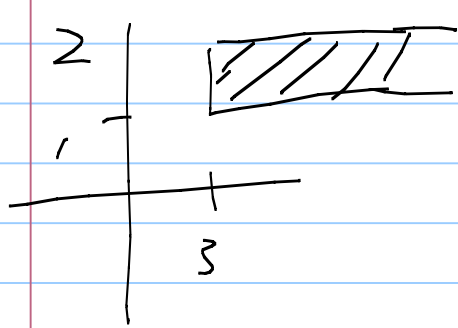
or $y=5$ is on the boundary of

the cases of $F_{XY}(x, y)$

Q: $P(1 < X \leq 2, 3 < Y \leq 5) = ?$

A:  $= F_{XY}(2, 5) - F_{XY}(2, 3) - F_{XY}(1, 5) + F_{XY}(1, 3)$

Q: $P(3 < X, 1 < Y \leq 2)$

 Ans: $F_{XY}(\infty, 2) - F_{XY}(3, 2) - F_{XY}(\infty, 1) + F_{XY}(3, 1)$

Q: $P(3 < X, 1 < Y < 2)$

A: $F_{XY}(\infty, 2^-) - F_{XY}(3, 2^-) - F_{XY}(\infty, 1) + F_{XY}(3, 1)$

* From joint cdf to marginal cdf

$F_X(x) = P(X \leq x) = F_{XY}(x, \infty)$

$F_Y(y) = P(Y \leq y) = F_{XY}(\infty, y)$

* From joint cdf $F_{XY}(x, y)$
to joint pdf $f_{XY}(x, y)$

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(s, t) dt ds$$

$$f_{XY}(x, y) = \frac{d}{dx} \frac{d}{dy} F_{XY}(x, y)$$

* From joint cdf to marginal pdf

Route 1: joint cdf $\xrightarrow{\text{diff}^2}$ joint pdf
 $\xrightarrow{\text{integration}}$ marginal pdf

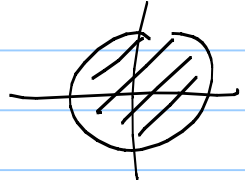
Route 2: joint cdf \rightarrow marginal cdf
 $\xrightarrow{\text{diff}}$ marginal pdf

HW11 Q11 Prob 5.18

(X, Y) is chosen uniformly randomly from a unit circle.

Q: $f_{XY}(x, y) = ?$

A: $f_{XY}(x, y) = \begin{cases} k & \text{if } \text{---} \text{---} \text{---} \\ 0 & \text{otherwise} \end{cases}$

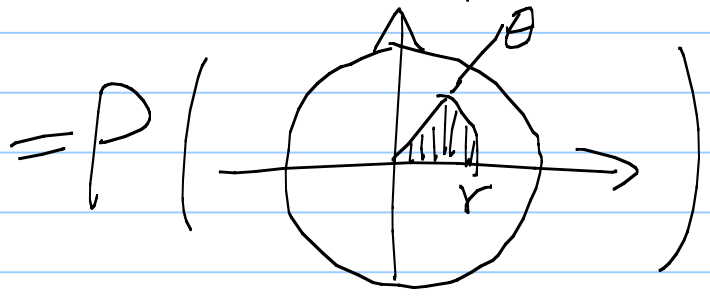


$\Rightarrow k \cdot \pi = 1 \quad k = \frac{1}{\pi}$

Q: Let $R = \sqrt{X^2 + Y^2}$
 $\Theta =$ the angle Q: Find $f_{R, \Theta}(r, \theta)$

Find $F_{R, \Theta}(r, \theta)$ the cdf.

Ans: $F_{R, \Theta}(r, \theta) = P(R \leq r, \Theta \leq \theta)$



$= \left(\frac{1}{2} r^2 \theta\right) \times k \quad \parallel \text{ Note: } k = \frac{1}{\pi}$

$$= \begin{cases} 0 & \text{Case 1 if } r < 0 \text{ or } \theta < 0 \\ \frac{1}{2} r^2 \theta - \frac{1}{\pi} & \text{Case 2 if } 0 \leq r < 1, \\ & 0 \leq \theta < 2\pi \\ r^2 & \text{Case 3 if } 0 \leq r < 1 \\ & 2\pi \leq \theta \\ \frac{\theta}{2\pi} & \text{Case 4 if } 1 \leq r \\ & 0 \leq \theta < 2\pi \\ 1 & \text{Case 5 if } 1 \leq r \\ & 2\pi \leq \theta \end{cases}$$

(b) $F_R(r) = ?$ the marginal cdf

$F_\Theta(\theta) = ?$ the marginal cdf

Ans: $F_R(r) = F_{R,\Theta}(r, \infty)$

$$= \begin{cases} 0 & \text{if } r < 0 \text{ (Case 1)} \\ r^2 & \text{if } 0 \leq r < 1 \text{ (Case 3)} \\ 1 & \text{if } 1 \leq r \text{ (Case 5)} \end{cases}$$

$F_\Theta(\theta) = F_{R,\Theta}(\infty, \theta)$

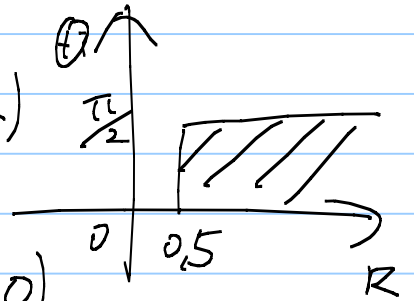
$$= \begin{cases} 0 & \text{if } \theta < 0 \text{ Case 1} \\ \frac{\theta}{2\pi} & \text{if } 0 \leq \theta < 2\pi \text{ Case 4} \\ 1 & \text{if } 2\pi \leq \theta \text{ Case 5} \end{cases}$$

Q: Marginal pdf? $f_R(r)$, $f_\Theta(\theta)$

164

Q3: $P(R > 0.5, 0 < \Theta \leq \frac{\pi}{2})$

$$= F_{R,\Theta}(\infty, \frac{\pi}{2}) - F_{R,\Theta}(0.5, \frac{\pi}{2}) - F_{R,\Theta}(\infty, 0) + F_{R,\Theta}(0.5, 0)$$



$$= \frac{(\frac{\pi}{2})}{2\pi} - \frac{1}{2}(0.5)^2 \frac{\pi}{2} / \pi$$

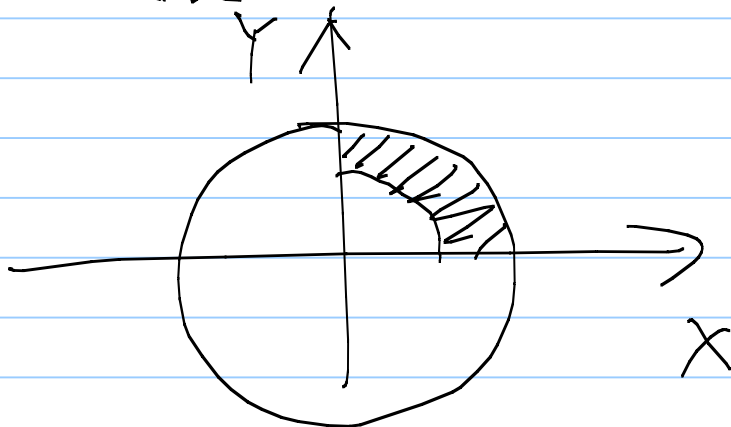
Case 4 Case 2

$$- \frac{\theta}{2\pi} - \frac{1}{2}(0.5)^2 \times 0 / \pi$$

Case 4 Case 2

$$= \frac{3}{16}$$

Alternative



$$= \frac{3}{16}$$

$$= \frac{1}{\pi} \left(\pi \cdot 1^2 \times \frac{1}{4} - \pi \times (0.5)^2 \times \frac{1}{4} \right)$$