

Chapter 5 Pairs of R.Vs.

Consider a R.V X with sample space $S_X = \{0, 1\}$ and another R.V Y with $S_Y = \{0, 1, 2\}$.

To discuss the joint relationship of X and Y , we need to consider the joint sample space

$$S_{XY} = \left\{ (0, 0)^{\frac{1}{6}}, (0, 1)^{\frac{1}{6}}, (0, 2)^{\frac{1}{6}}, (1, 0)^{\frac{1}{4}}, (1, 1)^0, (1, 2)^{\frac{1}{4}} \right\}$$

Once the W.A of S_{XY} is made, we can compute any prob like

$$P(X^2 \leq \sqrt{Y}), P(\max(X, Y) \leq 3) \dots$$

Note that Even $E(X^2 Y)$

it is no different than considering a Random Vector $W = (X, Y)$ where the output of each W is a vector.

Nonetheless, it is not efficient to always

assign the Weight for the joint sample space as most of the time, we

are interested only in $P(X > 0)$ $P(\sqrt{X} < 3)$

$E(X^{\frac{3}{2}})$, In most cases, we do the W.A gradually, start from the W.A of X , then extend to (X, Y)

We say the W.A for 'X' alone is

the marginal distribution.

	Y	0	1	2	
X	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
	1	$\frac{1}{4}$	0	$\frac{1}{4}$	
		$\frac{5}{12}$	$\frac{2}{12}$	$\frac{5}{12}$	

marginal distribution of X

marginal distribution

Marginal distribution

joint distribution

conditional distribution

marginal + conditional

joint

the key concept of every

computation

Note 1: marginal + marginal $\not\Rightarrow$ joint. 138

Ex

		Y				
	X		0	1	2	
	0	$\frac{5}{24}$	$\frac{2}{24}$	$\frac{5}{24}$		$\frac{1}{2}$
	1	$\frac{5}{24}$	$\frac{2}{24}$	$\frac{5}{24}$		$\frac{1}{2}$
		$\frac{5}{12}$	$\frac{2}{12}$	$\frac{5}{12}$		

has the same marginal distributions as the first example, but different joint distribution.

Ex HW10Q8

X is geometric with p.

↳ The marginal distribution

Given $X=x_0$, Y is a Poisson with

$$\alpha = x_0$$

↳ The conditional distribution

Q: The joint sample space = ?

Ans: $S_{XY} = \{$ all pairs of non-negative integers
 $(0,0), (0,1), (0,2) \dots$
 $(1,0), (1,1) \dots$
 \vdots

Q: What is the joint W.A ?

$$P(X=k, Y=h)$$

$$= \underbrace{P(Y=h | X=k)}_{\text{joint}} \cdot \underbrace{P(X=k)}_{\text{marginal}}$$

(think about the tree method)



$$= \begin{cases} \left(\frac{k^h}{h!} e^{-k} \right) \cdot p(1-p)^k & \text{if } 0 \leq h \\ & 0 \leq k \\ 0 & \text{otherwise} \end{cases}$$

Q: What is the marginal W.A of X ?

A: Summing over all Y values

$$P(X=k) = \sum_{h=0}^{\infty} P(X=k, Y=h) \stackrel{=1}{=} \begin{cases} p(1-p)^k \underbrace{\sum_{h=0}^{\infty} \frac{k^h}{h!} e^{-k}}_{=1} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

Q: How to compute the marginal distribution of Y ?

Ans: Summing over X 's

$$P(Y=h) = \sum_{k=0}^{\infty} P(X=k, Y=h)$$

$$= \sum_{k=0}^{\infty} \frac{k^h}{h!} e^{-k} p(1-p)^k$$

$$= \frac{p}{h!} \sum_{k=0}^{\infty} k^h e^{-k} (1-p)^k \quad \text{if } h \geq 0$$

$$0 \quad \text{if } h < 0$$

The computation of the sum is not easy.

$$\text{Ex: } P(Y=0) = \frac{p}{0!} \sum_{k=0}^{\infty} (k)^0 e^{-k} (1-p)^k$$

$$= p \times \frac{1}{1 - e^{-1}(1-p)}$$

$$P(Y=1) = \frac{p}{1!} \sum_{k=0}^{\infty} k e^{-k} (1-p)^k$$

$$= \frac{p}{1!} \times \frac{e^{-1}(1-p)}{(1 - e^{-1}(1-p))^2}$$

* The geometric series formulas

* But the concept is straightforward

$$Q: P(X^2 + Y^2 \leq 4)$$

$$\text{Ans: } P\left(\begin{array}{l} (0,0), (0,1), (0,2) \\ (1,0), (1,1) \\ (2,0) \end{array} \right) = 6 \text{ terms.}$$

Independence

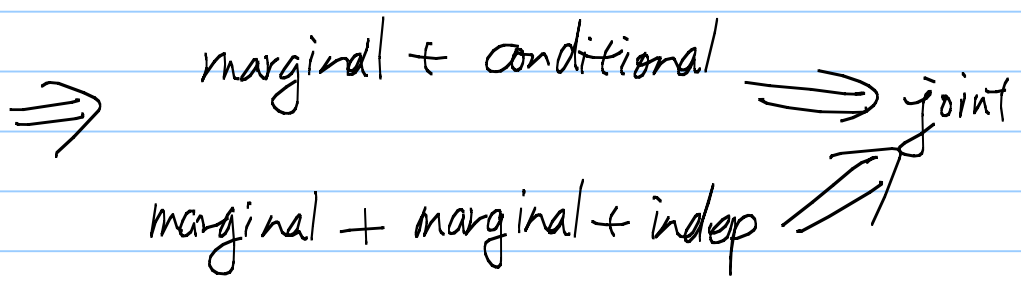
Two (marginal) R.Vs X and Y are independent. If their joint weight assignment is the product of the marginal probabilities.

That is

P(X=k, Y=h) = P(X=k) * P(Y=h)

Comparison, if not independent

P(X=k, Y=h) = P(X=k) * P(Y=h | X=k)



Note: Relate it to the tree/table method we have learned so far.

* Discrete 2-dim R.V.s.

(142)

$$S_{XY} = \left\{ \begin{array}{l} (-1, 0), (-1, 1) \dots \\ (0, 0), (0, 1) \\ (1, 0), \dots \end{array} \right.$$

all grid points $\}.$

The Joint W.A is specified by the joint pmf: $P_{k,h} = P(X=k, Y=h)$

s.t. $P_{k,h} \geq 0$

$$\sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} P_{k,h} = 1.$$

* From joint distribution to marginal distri:

$$P(Y=h) = \sum_{k=-\infty}^{\infty} P_{k,h}$$

focusing on the column

summing over
uninterested
variable

* Expectation of a function of a 2-dim R.V

$$E(f(X, Y)) \quad \text{say } f(x, y) = x^2 + y^2$$

$$\text{is } \sum_k \sum_h f(k, h) \underbrace{P_{k,h}}_{\text{weight}}$$

$$E(X^2 + Y^2 + XY)$$

$$Q: E(U(1-X-Y))$$

$$U(1-x-y) = \begin{cases} 1 & \text{if } 1-x-y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Ans} = 1 \times P_{0,0} + 1 \times P_{0,1} + 1 \times P_{1,0} + 0 \dots$$

= 3 terms

HW10Q8

X is geometric with p.

↳ The marginal distribution

Given $X = \lambda_0$, Y is a Poisson with

$$\lambda = \lambda_0$$

↳ The conditional distribution

Q: $E(X) = ?$

Ans: $\frac{1-p}{p}$ by table look-up since the (marginal) distribution of X is geometric

Q: $E(Y) = ?$

Ans: We can compute either

① $\sum_{y=0}^{\infty} y P(Y=y)$ hard, we don't even know $P(Y=y)$

Or ② $\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} y P(X=x, Y=y) = \sum_{y=0}^{\infty} y \sum_{x=0}^{\infty} P(X=x, Y=y)$

The difference is how you compute the weighted average, in a "① Column by column" way or "② block by block" way

from ② $\Rightarrow \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} y \cdot \frac{x^y}{y!} e^{-x} p(1-p)^x$

$$= \sum_{X=0}^{\infty} \underbrace{X}_{\text{by table look up for the expected value of a Poisson}} p(1-p)^X$$

$$= \frac{1-p}{p} \quad \text{by another table look up.}$$

Summary: For 2-dim R.V.s, there are many ways of "counting the weights". Some are easier than the others.

* Conditional distribution for discrete variables.

$$P(X=k | Y=h) \quad \text{focusing on the } h\text{-th col.}$$

$$= \frac{P(X=k, Y=h)}{P(Y=h)}$$

$$= \frac{P_{k,h}}{\sum_{x=-\infty}^{\infty} P_{x,h}}$$

Exercise: $P(Y=h | X=k) = \frac{P_{k,h}}{\sum_{y=-\infty}^{\infty} P_{k,y}}$

Summary

Note Title

* Discrete 2-dim R.V.s.

$$S_{XY} = \{ \text{all grid points} \}$$

W.A: $p_{k,h} = P(X=k, Y=h)$ (2-dim) joint pmf

Expectation $E(g(X, Y))$

$$= \sum_k \sum_h g(k, h) p_{k,h}$$

Marginal pmf $P(X=k) = p_k = \sum_{h=-\infty}^{\infty} p_{k,h}$

Summing over the row of interest.

$$P(Y=h) = p_h = \sum_{k=-\infty}^{\infty} p_{k,h}$$

Conditional pmf $p_{k|Y=h} = P(X=k | Y=h)$

$$\triangleq \frac{p_{k,h}}{\sum_{x=-\infty}^{\infty} p_{x,h}}$$

Conditional expectation ?

Conditional pmf + expectation.

* Continuous 2-dim Random variables.

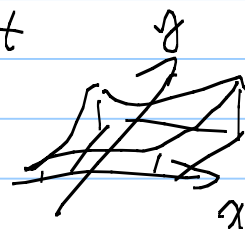
$S_{XY} = \{ \text{all pairs of real numbers} \\ (-\infty, \infty) \times (-\infty, \infty) \}$

say $(0.1, 1001.5), (\pi, e) \dots$

The joint W.A is specified by the joint

pdf: $f_{XY}(x, y)$ such that

$$f_{XY}(x, y) \geq 0$$



$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

$$P((X, Y) \in A)$$

\hookrightarrow falls in/belongs to

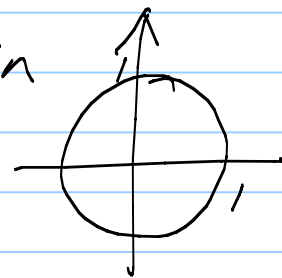
$$= \int_A f_{XY}(x, y) dx dy$$

Problem 5.28(ii)

Ex: HW11Q3. A joint pdf is given

as follows $f_{XY}(x, y) = \begin{cases} k & \text{if } (x, y) \text{ in} \end{cases}$

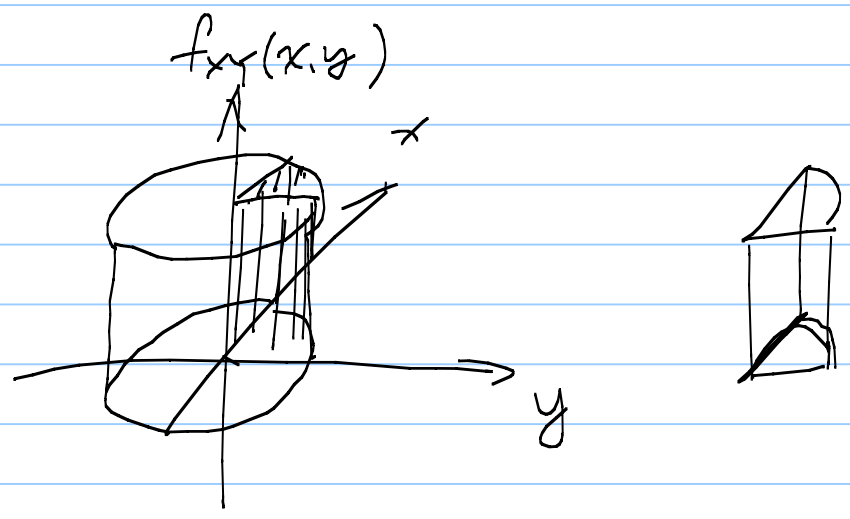
0 otherwise.



Q: find the k value.

$$\Rightarrow k \times \pi \times r^2 = 1 \Rightarrow k = \frac{1}{\pi}$$

Q⁹ Find $P(X > 0, Y > 0)$?



$$\text{Ans} = k \times \left(\frac{1}{4} \pi r^2\right) = \frac{1}{4}$$

* Another example.

Ex: $f_{xy}(x,y) = \begin{cases} kxy & \text{if } 0 < x < 1 \\ & \text{and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

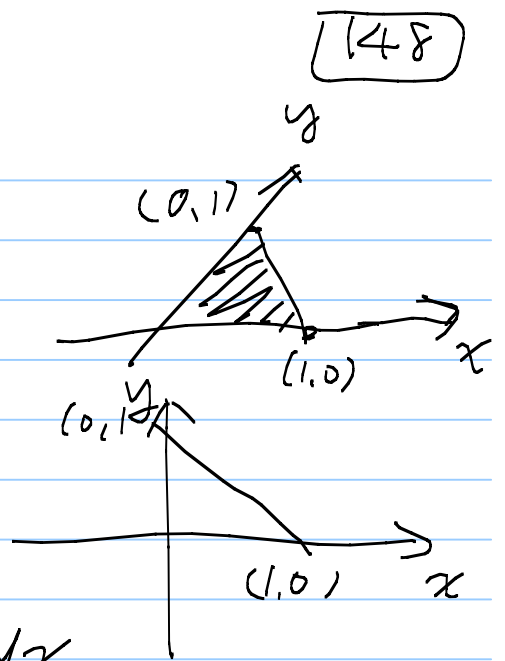
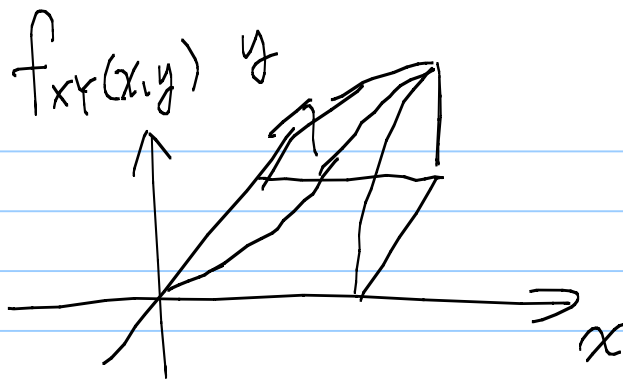
is a joint pdf, find k value.

② $P(X+Y \leq 1)$

$$\text{Ans: } \int_0^1 \int_0^1 kxy \, dx \, dy = 1$$

$$\Rightarrow \int_0^1 ky \left(\frac{x^2}{2} \Big|_0^1\right) dy \Rightarrow \boxed{k=4}$$

$$= \int_0^1 \frac{1}{2}ky \, dy = \frac{1}{2}k \left(\frac{y^2}{2} \Big|_0^1\right) = \frac{k}{4}$$



Q2:

$$P(X+Y \leq 1) = \int_{x=0}^1 \int_{y=0}^{1-x} 4xy \, dy \, dx$$

$$= \int_{x=0}^1 4x \left(\frac{y^2}{2} \Big|_0^{1-x} \right) dx$$

$$= \int_{x=0}^1 4x \cdot \frac{(1-x)^2}{2} dx$$

$$= \int_{x=0}^1 2x - 4x^2 + 2x^3 dx$$

$$= x^2 - \frac{4}{3}x^3 + \frac{2}{4}x^4 \Big|_0^1 = \frac{1}{6} \quad \#$$

Exercise $P(X+Y \leq 1.5) = ?$

Ans: $\iint_{\square} 4xy \, dy \, dx$

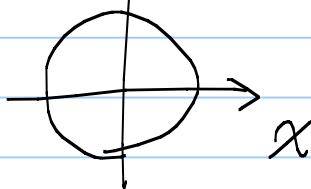
$$\begin{aligned}
 &= \iint_{\square} 4xy \, dy \, dx + \iint_{\square} 4xy \, dy \, dx \\
 &= \int_0^{0.5} \int_0^1 4xy \, dy \, dx + \int_{0.5}^1 \int_0^{1-x} 4xy \, dy \, dx \\
 &=
 \end{aligned}$$

* From joint pdf to marginal pdf.

Knowing $f_{XY}(x, y)$ how to construct
 $f_X(x)$ (or $f_Y(y)$)

Ans: Integrating over uninterested variable.

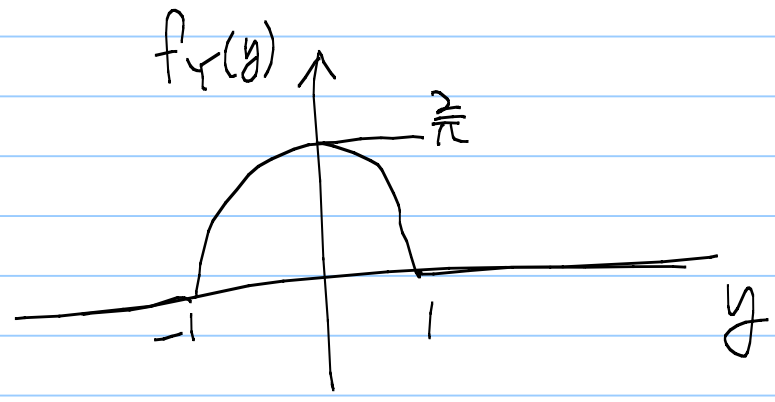
Ex: HW11 Q3 Problem 5.28(i) of

$$f_{XY}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } \text{circle} \\ 0 & \text{otherwise.} \end{cases}$$


Find the marginal $f_Y(y)$.

Ans: $f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y) dx$

$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx$ if $y < -1$
 if $-1 \leq y < 1$
 $= \frac{2\sqrt{1-y^2}}{\pi}$
 0 if $1 \leq y$



Joint pdf \rightarrow marginal pdf \checkmark

How about marginal + conditional pdf

\rightarrow joint pdf?