Using cdf to find new pdf is very important. One more example:

HW6Q12

Q: X is chosen uniformly from $(0,1)$

$$Y = \frac{-\ln(x)}{\lambda} \quad \text{for some } \lambda > 0$$

Find out the cdf & the pdf of Y.

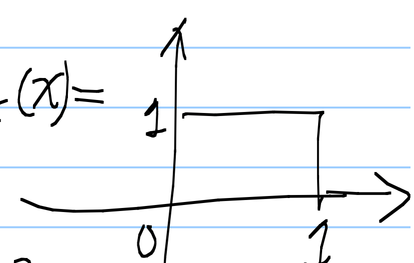
Ans: $F_Y(y) \triangleq P(Y \le y)$

$$= P\left(\frac{-\ln(X)}{\lambda} \le y\right)$$

$$= P(+\ln(X) \ge -\lambda y)$$

$$= P(X \ge e^{-\lambda y})$$

$$= \int_{e^{-\lambda y}}^{\infty} f_X(x)\,dx.$$

$\Big\|$ $f_X(x) =$ 

Case 1: If $y<0$ (then $e^{-\lambda y} > 1$)

$F_Y(y) = 0$

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If $y \geq 0$ $(e^{-\lambda y} \leq 1)$

$$F_Y(y) = \int_{e^{-\lambda y}}^{1} 1 \, dx = 1 - e^{-\lambda y}$$

pdf: $f_Y(y)$?

$$= \frac{d}{dy} f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \lambda e^{-\lambda y} & \text{if } y \geq 0. \end{cases}$$

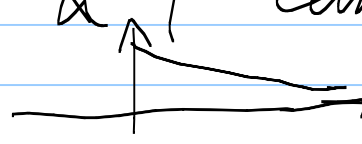
Q: What type of R.V.s is Y ?

Ans: Y is an exponential R.V.

Q: Why this is an important question?

Ans: Computer knows how to generate a uniform R.Vs. X between $(0,1)$ ex: "rand()" in MATLAB.

By taking $Y = -\frac{\ln(X)}{\lambda}$, the Y is an exponential R.V with more Y's closer to zero, & Y can be extremely large



Advantage

⑤ For positive $\left\{ \begin{matrix} \text{discrete} / \text{conti} \end{matrix} \right.$ R.V.s (those X with $P(X<0)=0$)

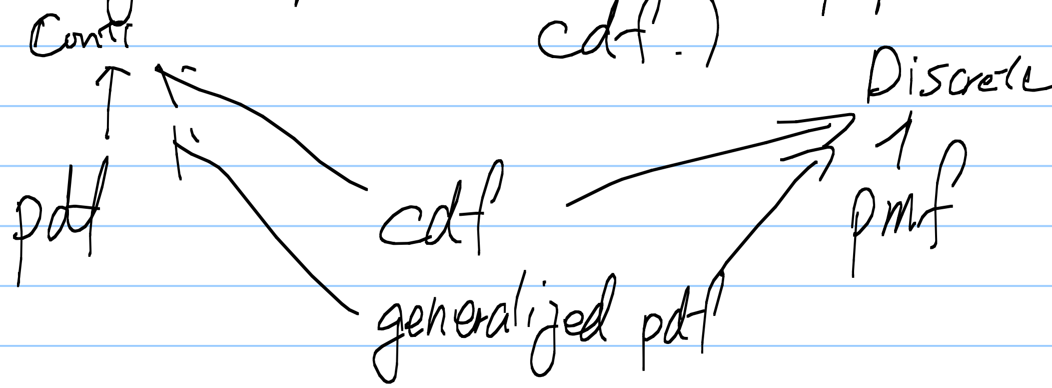
$$E(X) = \int_0^\infty (1 - F_X(x))\,dx$$

Ex: The $F_X(x)$ of an exponential R.V. is $\boxed{1 - e^{-\lambda x}}$

$$\Rightarrow E(X) = \int_0^\infty (1 - (1 - e^{-\lambda x}))\,dx$$

$$= \int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda} \quad \#$$

* Generalized pdf: (Not as popular as cdf.)



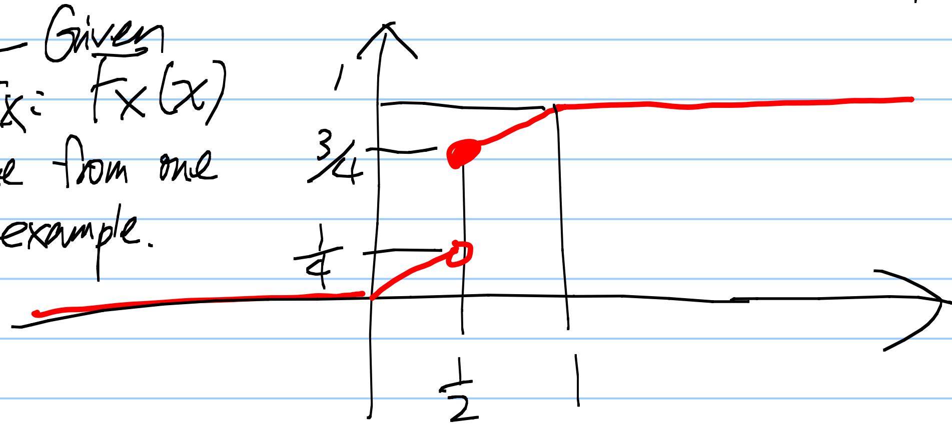
Recall $f_X(x) = \frac{d}{dx} F_X(x)$

The disadvantage is that those jumps are not differentiable.

$\Longrightarrow$ Introduce the $\delta(x)$ impulse function

Ex: Given $F_X(x)$

Continue from one previous example.

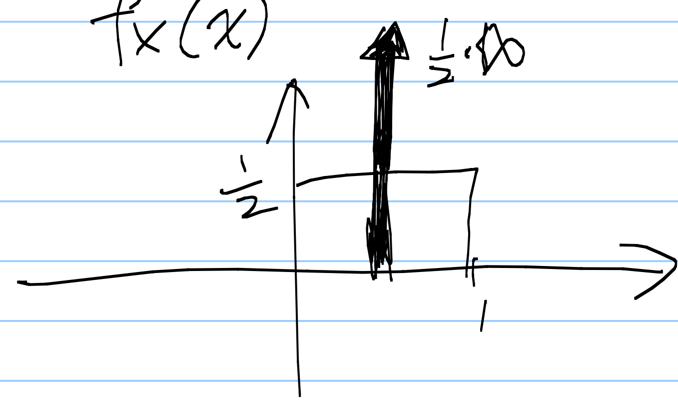


Q: What is the generalized pdf.

Ans: Still do differentiation, and add $\Delta\,\delta(x-s)$ for a Δ-jump at $x = s$.

$$f_X(x) = \frac{1}{2}\delta(x-\tfrac{1}{2}) + \begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2} & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \le x < 1 \\ 0 & \text{if } 1 \le x \end{cases}$$

Plot $f_X(x)$



Recall the experiment that corresponds to the above derivation:

> Flip a coin, if head $X = \frac{1}{2}$
> if tail, X is chosen randomly between $(0,1)$

Q: Can we directly derive the generalized pdf?

Ans: ① With prob $\frac{1}{2}$ (head), the output will be "exactly" $\frac{1}{2}$.

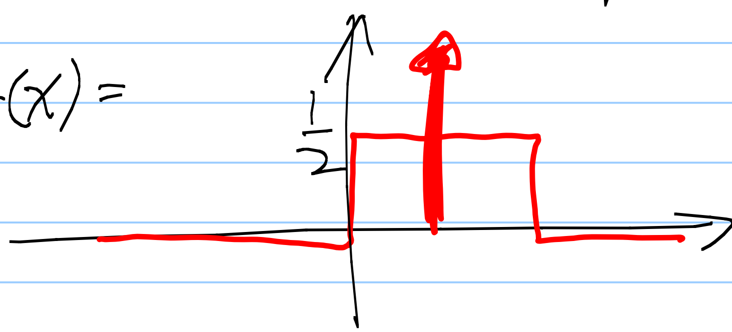
$\Rightarrow$ We need a "$\frac{1}{2}\delta(x-\frac{1}{2})$" term for the discrete prob mass function.

② for the remaining case X is chosen uniformly from (0, 1)

We need $f_X(x) = \frac{1}{2} \times \begin{cases} 1 & \text{when } 0<x<1 \\ 0 & \end{cases}$

$$\Rightarrow f_X(x) = \frac{1}{2}\delta(x-\frac{1}{2}) + \begin{cases} \frac{1}{2} & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$f_X(x) =$

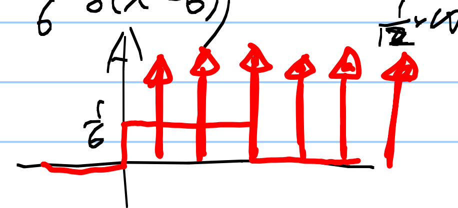


The result is the same as if we derive $f_X(x)$ from $F_X(x)$.

Ex: Flip a fair coin, if head, X is the outcome of a fair dice
if tail, X is chosen uniformly random from (0,3). Find the pdf ~~of~~ X.

Ans: $f_X(x) = \frac{1}{2}\left(\frac{1}{6}\delta(x-1) + \frac{1}{6}\delta(x-2) + \cdots \frac{1}{6}\delta(x-6)\right)$

$$+ \frac{1}{2} \times \begin{cases} \frac{1}{3} & \text{if } 0<x<3 \\ 0 & \text{otherwise} \end{cases}$$



HW6 Q10

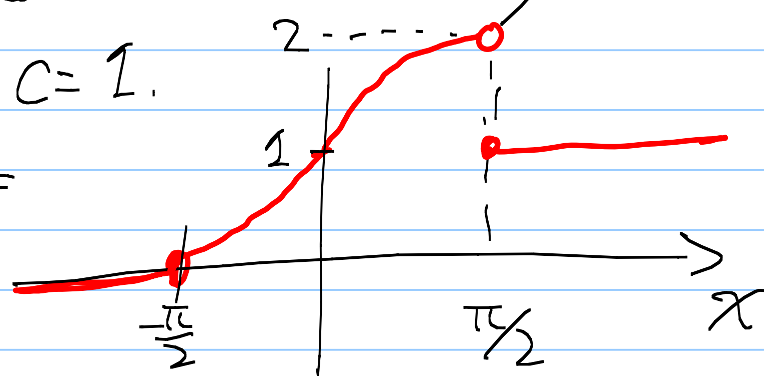
$F_X(x)$ is a cdf of X, and we know that

$$F_X(x) = \begin{cases} 0 & x < -\frac{\pi}{2} \\ c(1+\sin(x)) & -\pi/2 \le x < \pi/2 \\ 1 & \pi/2 \le x \end{cases}$$

Q1: c cannot be 1. Why?

Ans: If $c = 1$.

$F_X(x) =$



It violates ④

Q2 If $c = \frac{1}{2}$. Show ~~that~~ X is a conti R.V.

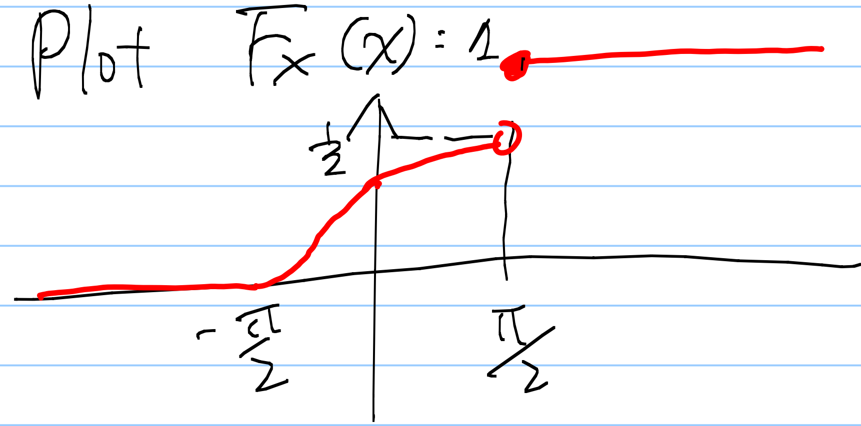
Ans: $F_X(x) =$

is conti (no jump)

$\Rightarrow X$ is a conti R.V.

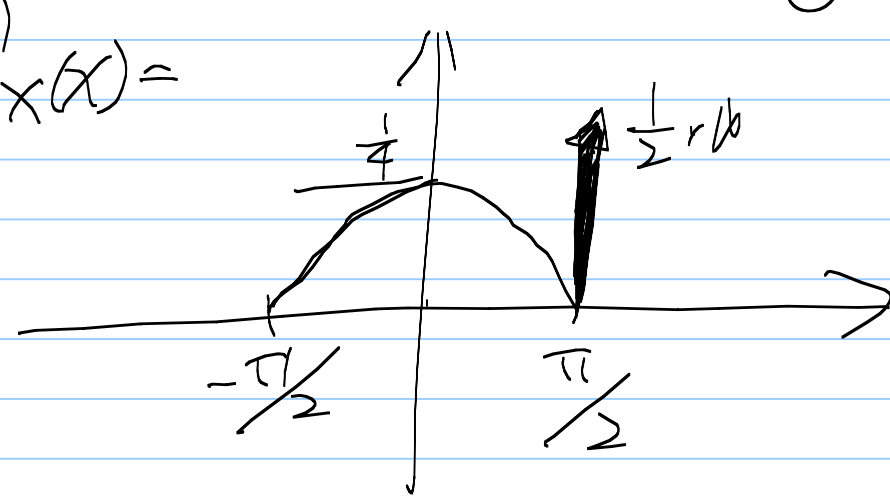
Q3: Let $c = \frac{1}{4}$, Find the generalized pdf of X

Ans: Plot $F_X(x)$:



Ans: $f_X(x) = \underline{\frac{1}{2}\delta(x-\frac{\pi}{2})} + \begin{cases} 0 & \text{if } x < -\frac{\pi}{2} \\ \frac{d}{dx}\left(\frac{1}{4}(1+\sin(x))\right) = \frac{1}{4}\cos(x) & \text{if } -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x \end{cases}$

$f_X(x) =$



Many of the existing concept can be combined.

* Combining generalized pdf with expectation

* In the previous example, we consider the following exp. Flip a fair coin, if head, X is the outcome of a fair dice

if tail, X is chosen uniformly random from $(0,3)$.

New question: Find $E(X)$ and $Var(X)$

Ans: $E(X) = \int_{x=-\infty} x f_X(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{x}{12}\left(\delta(x-1) + \cdots + \delta(x-6)\right) dx$$

$$+ \int_{-\infty}^{\infty} \frac{x}{2}\left(\begin{cases} \frac{1}{3} & \text{if } 0<x<3 \\ 0 & \text{otherwise}\end{cases}\right) dx$$

$$= \frac{1}{12}(1+2+3+4+5+6)$$

$$+ \int_0^3 \frac{x}{6}\,dx$$

$\because \int_{-\infty}^{\infty} x\,\delta(x-a)\,dx = a$

$$= \frac{7}{4} + \frac{3}{4} = \frac{10}{4} = \frac{5}{2}$$

$Var(X)$. exercise

Combining conditional prob with cdf.

* Conditional cdf

Definition:

$$F_X(x \mid a \leq X \leq b)$$

$$\triangleq P(X \leq x \mid a \leq X \leq b)$$

$$= \frac{P(X \leq x \text{ and } a \leq X \leq b)}{P(a \leq X \leq b)}$$

Example: X is a fair dice

What is the conditional cdf given $0.3 \leq X \leq 4.01$.

Ans: $F_X(x) = P(X \leq x \mid 0.3 \leq X \leq 4.01)$

$$= \frac{P(X \leq x \text{ and } 0.3 \leq X \leq 4.01)}{P(0.3 \leq X \leq 4.01)}$$

$$= \begin{cases} \dfrac{0}{4 \times \frac{1}{6}} & \text{if } x < 1 \\ \dfrac{P_1}{4 \times \frac{1}{6}} = \dfrac{\frac{1}{6}}{4 \times \frac{1}{6}} & \text{if } 1 \leq x < 2 \\ & 2 \leq x < 3 \\ & 3 \leq x < 4 \\ & 4 \leq x \end{cases}$$

Conditional pmf:

$$p(X=k \mid a \le X \le b) = \frac{p_k}{\sum_{a \le s \le b} p_s}$$

Conditional pdf: $f_X(x \mid a \le X \le b)$

Solution 1: differentiate the conditional cdf

$$f_X(x \mid a \le X \le b) = \frac{d}{dx} F_X(x \mid a \le X \le b)$$

Solution 2:

$$f_X(x \mid a \le X \le b) = \begin{cases} 0 & \text{if } x \text{ is not between } a \le x \le b. \\ c \cdot f_X(x) & \text{if } x \text{ is in } a \le x \le b \end{cases}$$

where $f_X(x)$ is the original pdf.

We just need to compute the normalization coefficient c & make sure

$$\int_a^b c f_X(x)\,dx = 1 \text{ \#}$$

$$\Leftrightarrow \quad c = \frac{1}{\int_a^b f_X(s)\,ds}$$

* Conditional Expetation / Variance

Note Title 2/23/2011

Combination of conditional prob, pmf/pdf, and expetation

Ex: X is a unfair dice. with W.A

1	2	3	4	5	6
$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

Conditioning on $0.3 \leq X \leq 4.01$

What is the conditional expectation of X, What is the conditional variance

Ans: Step 1: Find the conditional pdf/pmf

(Sometimes, you may need to start from the conditional cdf)

Step 2: Compute the weighted average using the conditional pmf/pdf.

Step 1: $P(X = k \mid 0.3 \leq X \leq 4.01)$

$$= \begin{cases} 0 & \text{if } k \leq 0 \text{ or } k \geq 5 \\ \dfrac{p_k}{p_1 + p_2 + p_3 + p_4} & \text{if } 1 \leq k \leq 4 \end{cases}$$

$$= \begin{cases} \frac{2}{5} & \text{if} \quad k=1 \\ \frac{1}{5} & \text{if} \quad k=2,3,4 \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: Conditional expectation

$$E(X \mid 0.3 \le X \le 4.01)$$

$=$ conditioning on

$$= 1 \times \frac{2}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} + 4 \times \frac{1}{5}$$

$$= \frac{11}{5} \ \text{※}$$

$$Var(X \mid 0.3 \le X \le 4.01)$$

$$= E(X^2 \mid 0.3 \le X \le 4.01) - \left(\frac{11}{5}\right)^2$$

$$= 1^2 \times \frac{2}{5} + 2^2 \times \frac{1}{5} + 3^2 \times \frac{1}{5} + 4^2 \times \frac{1}{5} - \left(\frac{11}{5}\right)^2$$

$$= \left(\frac{31}{5}\right) - \left(\frac{11}{5}\right)^2 = \frac{34}{25}$$

Note Title 2/25/2011

Other (unifying) descriptions of a R.V. ① Characteristic function of a R.V.

Why so many ways to describe a W.A.? Mathematicians are hoping that by using a different way of describing the same W.A., the counting part can be easier

② Moment generating function of a R.V. ③ prob. generating function ④ cdf, ⑤ generalized pdf(s)

* Characteristic function of a R.V.

$\Phi_X(\omega)$ is a function of a parameter ω

$$\Phi_X(\omega) \triangleq E\left(e^{j\omega X}\right)$$

Ex: X is a Bernoulli R.V with para $p = \frac{1}{\pi}$

Find the characteristic function $\Phi_X(\omega)$.

Ans: $\Phi_X(\omega) = E\left(e^{j\omega X}\right)$

$$= e^{j\omega \cdot 0} \times \left(1 - \frac{1}{\pi}\right) + e^{j\omega \cdot 1}\left(\frac{1}{\pi}\right)$$

$$= \left(1 - \frac{1}{\pi}\right) + \frac{1}{\pi} e^{j\omega} \quad \#$$

Ex: X is a binomial R.V w. para

$n=20 \quad p=0.7$

Find $\Phi_X(\omega)$.

Before solving $\Phi_X(\omega)$, We need the binomial theorem

(p.61)

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{(n-k)}}$$

ex: $a=\pi$

$b=0.3$

Why ↳ holds?

$\because$ We know $1 = \sum_{k=0}^{\infty} \binom{n}{k} p^k (1-p)^k$

for all $0<p<1$.

Let $p = \frac{a}{a+b}$

$$\therefore \quad 1 = \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k}$$

$$\therefore (a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^k$$

Ans: $\Phi_X(\omega) = E(e^{j\omega X})$

$$= \sum_{k=0}^{n} e^{j\omega k} \binom{n}{k} p^k (1-p)^{(n-k)}$$

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$$= \sum_{k=0}^{n} \binom{n}{k} \underbrace{(e^{j\omega} p)^k}_{a} \underbrace{(1-p)^{n-k}}_{b}$$

$$= (a+b)^n = (e^{j\omega} p + (1-p))^n$$ ※

What if X is a conti R.V.

$$\Phi_X(\omega) = E(e^{j\omega X})$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx$$

$$= \underline{\underline{\int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx}}.$$

Is similar to the ~~Fourier transform of $f_X(\omega)$~~

$$\Rightarrow f_X(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

Ex: X is exponential R.V with para λ

Find $\Phi_X(\omega)$?

Ans: $\Phi_X(\omega) = E(e^{j\omega X})$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_{0}^{\infty} e^{j\omega x} \lambda e^{-\lambda x}\, dx$$

$$= \frac{\lambda}{\lambda - j\omega} \quad \#$$

How to use $\Phi_X(\omega)$?

Ans: One way ~~of~~ using $\Phi_X(\omega)$ is the moment theorem

* For any discrete / conti / mixed type R.V X with $\Phi_X(\omega)$

$E(X^n)$ the n-th moment

$$= \left(\frac{1}{j^n}\right) \underbrace{\left[\frac{d^n}{d\omega^n} \Phi_X(\omega)\right]_{\omega=0}}$$

Step1: differentiate it n times.

Step2: then evaluate the value by plugging $\omega=0$.

Ex: X has $\Phi_X(\omega) = \dfrac{2}{2 - j\omega}$

Q: What type of R.Vs is X?

Ans: X is exponential with para $\lambda = 2$

Now we have ① pdf/pmf ② cdf $\boxed{\text{③ Characteristic function}}$ to describe a R.V.

Q: $E(X) = ?$

Ans: $E(X) = \dfrac{1}{j^1} \left(\dfrac{d}{d\omega} \dfrac{2}{2 - j\omega} \right) \Bigg|_{\omega=0}$

$= \dfrac{1}{j} \times \dfrac{2}{(2 - j\omega)^2} \times (-j) \times (-1) \Bigg|$

$= \dfrac{1}{j} \times \dfrac{2j}{(2 - j\omega)^2} \Bigg|_{\omega=0} = \dfrac{1}{2} = \dfrac{1}{\lambda}$

Q: $E(X^2) = ?$

A $E(X^2) = \dfrac{1}{j^2} \left(\dfrac{d^2}{d\omega^2} \Phi_X(\omega) \right)_{\omega=0}$

$= \dfrac{1}{j^2} \left(\dfrac{d}{d\omega} \dfrac{2j}{(2 - j\omega)^2} \right)_{\omega=0}$

$= \dfrac{1}{j^2} \left((-2) \dfrac{2j}{(2 - j\omega)^3} \times (-j) \right) \Bigg|_{\omega=0}$

$$= \frac{1}{j^2} \left(\frac{4j^2}{(2-j\omega)^3} \right)_{\omega=0}$$

$$= \frac{4}{8} = \frac{1}{2}$$

Q: $Var(X) = ?$

Ans: $E(X^2) - \left(E(X)\right)^2$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} = \frac{1}{\lambda^2}$$

(see Table 4.1)

* Characteristic function

$$\Phi_X(\omega) = E\left(e^{j\omega X}\right)$$

Ex: X is Gaussian with μ, σ^2

Find $\Phi_X(\omega)$.

Ans: $\Phi_X(\omega) = E(e^{j\omega X})$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Again, it is the integration being the most difficult part.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 - 2\sigma^2 j\omega x)}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2(\mu + j\omega\sigma^2)x + (\mu + j\omega\sigma^2)^2 - (\mu + j\omega\sigma^2)^2 + \mu^2\right)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[(x - (\mu + j\omega\sigma^2))^2 - 2\mu j\omega\sigma^2 - (j\omega\sigma^2)^2\right]} dx$$

$$= \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x - (\mu + j\omega\sigma^2))^2}}_{=1} \times e^{+j\omega\mu - \frac{1}{2}\omega^2\sigma^2} dx$$

$$= e^{+j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

\#

Q $E(X) = ?$ $\quad E(X^2) = ?$ $\quad Var(X) = ?$

Ans:

① $\frac{d}{d\omega} \Phi_X(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2} (j\mu - \omega\sigma^2)$

② $\frac{d^2}{d\omega^2} \Phi_X(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2} \left((j\mu - \omega\sigma^2)^2 - \sigma^2 \right)$

$\Rightarrow E(X) = \frac{1}{j} \left(① \Big|_{\omega=0} \right) = \frac{1}{j} \cdot j\mu = \mu$

$E(X^2) = \left(\frac{1}{j}\right)^2 \left(② \Big|_{\omega=0} \right) = \mu^2 + \sigma^2$

$Var(X) = E(X^2) - \mu^2 = \sigma^2$

① Characteristic function

$$\Phi_X(\omega) = E(e^{j\omega X})$$

Moment theorem

$$E(X^n) = \left(\frac{1}{j^n}\right)\left[\frac{d^n}{d\omega^n}\Phi_X(\omega)\right]_{\omega=0}$$

$\Phi_X(\omega)$: Fourier transform of $f_X(x)$

② If we know that X is always non-negative $P(X<0)=0$, then we can get rid of the "j" by considering the following

$\boxed{\text{Moment generating function}}$

$X^*(s)$ is a function of a variable s

$$X^*(s) \triangleq E(e^{-sX})$$

Q: How to compute a moment generation function?

A: Simply evaluate $E(e^{-sX})$

118 Ex: Find the moment generation function of a Poisson R.V. w. para. α

Ans: $\sum_{k=0}^{n} \frac{\alpha^k}{k!} e^{-\alpha} \cdot e^{-sk} = \sum_{k=0}^{\infty} \frac{(\alpha e^{-s})^k}{k!} e^{-(\alpha e^{-s})} \cdot e^{-\alpha} \cdot e^{(\alpha e^{-s})} = e^{-\alpha(1-e^{-s})}$

$\ast$ One-way of using $X^{\ast}(s)$ is

$$E(X^n) = (-1)^n \left[\frac{d^n}{ds^n} X^{\ast}(s)\right]_{s=0}$$

$\boxed{\text{the moment theorem}}$

---

③ If we know that X is not only non-negative $P(X<0)=0$, but also outputs only integers, $S=\{0,1,2,\dots\}$. The we have

$\boxed{\text{probability generating function}}$

$G_X(z)$ is a function of variable z.

$$G_X(z) = E(z^X) = \sum_{k=0}^{\infty} z^k p_k$$

Ex: X is a fair dice

$G_X(z) = ?$

Ans: $G_X(z) = \sum_{k=1}^{6} z^k \frac{1}{6}$

$= z + z^2 + \cdots + z^6$

Q: How to use $G_X(z)$? Two possibilities.

Ans: ① $p_k = \frac{1}{k!}\left[\frac{d^k}{dz^k} G_X(z)\right]_{z=0}$