A question that is similar to HW5Q7.

$\boxed{029}$

Note Title 2/11/2011

Q: X is an exponential R.V w. para. λ

Show that for any $a, b > 0$.

$$P(X > a+b \mid X > a) = P(X > b)$$

Ans: RHS: $\int_b^\infty \lambda e^{-\lambda x} dx = e^{-b}$

LHS: $\dfrac{P(X > a+b)}{P(X > a)}$

$$= \frac{\int_{a+b}^\infty \lambda e^{-\lambda x} dx}{\int_a^\infty \lambda e^{-\lambda x} dx} = \frac{e^{-(a+b)}}{e^{-a}}$$

$$= e^{-b}$$

* It is called the $\boxed{\text{memoryless property.}}$ Why? Recall that X generally models the time you wait for the first customer.

Given that I have waited for a secs, the prob that I have to wait for additional b secs does not depend on how large/small a is. There is no memory to how long I have waited.

3. Gaussian / Normal R.V. w. para μ, σ.

$S = (-\infty, \infty)$ any real number

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

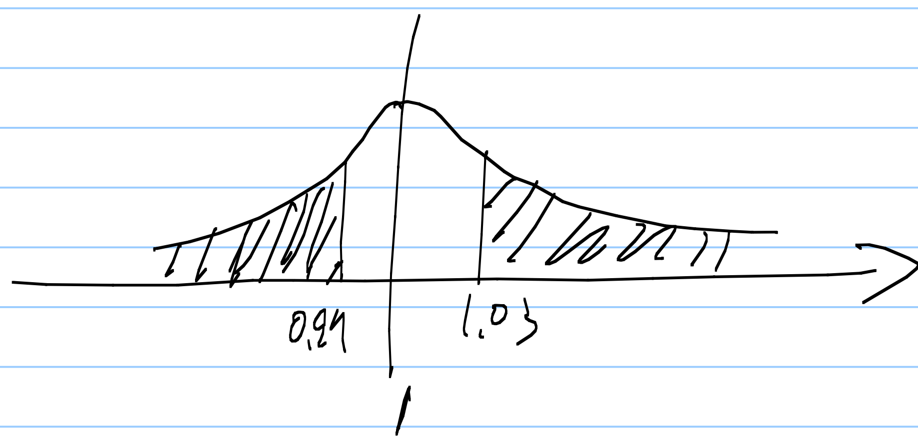
$$= \underbrace{\int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{\text{Odd function}}$$

$$+ \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= 0 + 1$$

$Var(X) = \sigma^2$ (hard to derive)

Ex: The reading of the GPS device X is a Gaussian R.V with $\mu = 1$ $\sigma = 0.01$ (where $\mu = 1$ is the actual location of the device



Q: What is the prob the GPS reading is 3 % off the actual location.

Ans: $P(X < 0.97 \text{ or } X > 1.03)$

$$= \int_{-\infty}^{0.97} \frac{1}{\sqrt{2\pi \times 0.01^2}} e^{-\frac{(x-1)^2}{2 \times 0.01^2}} dx$$

$$+ \int_{1.03}^{\infty} \frac{1}{\sqrt{2\pi \times 0.01^2}} e^{-\frac{(x-1)^2}{2 \times (0.01)^2}} dx$$

by MATLAB $\approx 0.27\%$

Other important conti R.Vs. include Laplacian, and Rayleigh R.Vs. See p.165 for their S and f_X description.

Discrete & Conti R.Vs are similar ex: $E(X+X^2) = E(X) + E(X^2)$

$$Var(X) = E((X-m)^2) = E(X^2) - m^2$$

How to use a unifying description to describe both types of R.V?

* Cumulative Distribution Function (cdf)

$\boxed{F_X(x)}$ where the input can be any real number ex: $x = 0.01$

$x = \pi \times 10^5$

$$\begin{tikzcd}
\text{Discrete R.V.} & & \text{Conti R.V} \\
\text{pmf } p_k \arrow[u] \arrow[r, shift left, "\mathrm{I}"] & \text{cdf } F(x) \arrow[ul] \arrow[ur] \arrow[l, shift left, "\mathrm{II}"] \arrow[r, shift right, "\mathrm{IV}"'] & \text{pdf } f_X(x) \arrow[u] \arrow[l, shift right, "\mathrm{III}"']
\end{tikzcd}$$

Definition

$$F_X(x) \triangleq P(X \le x).$$

$\boxed{\text{I}}$

* Discrete R.V. from p_k to $F_X(x)$

Ex: X is a bernoulli R.V. with $p = \frac{1}{3}$

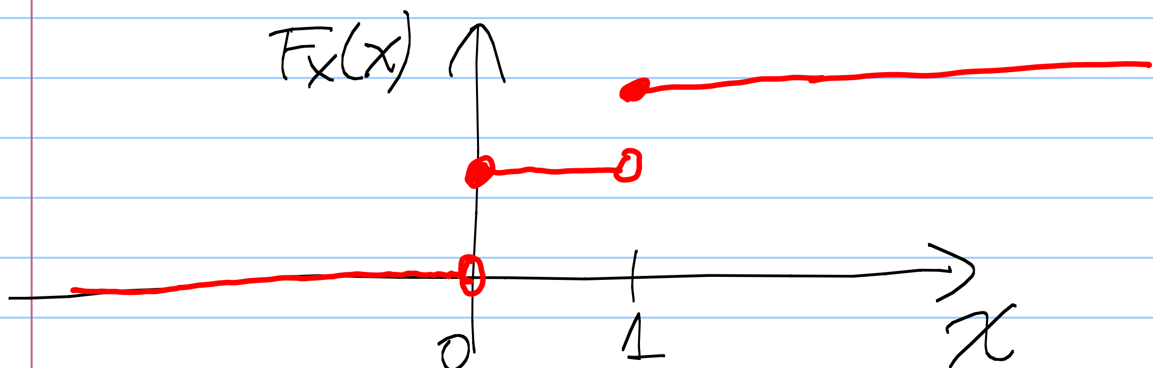
Find its cdf $F_X(x)$, Plot $F_X(x)$

Ans: $F_X(x) = P(X \le x)$ (Note that X is integer but x does not need to be integer)

$$F_X(\pi) = P(X \le \pi)$$

$$= \frac{2}{3} + \frac{1}{3} = 1.$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2/3 = p_0 & 0 \le x < 1 \\ \frac{2}{3} + \frac{1}{3} = p_0 + p_1 & 1 \le x \end{cases}$$

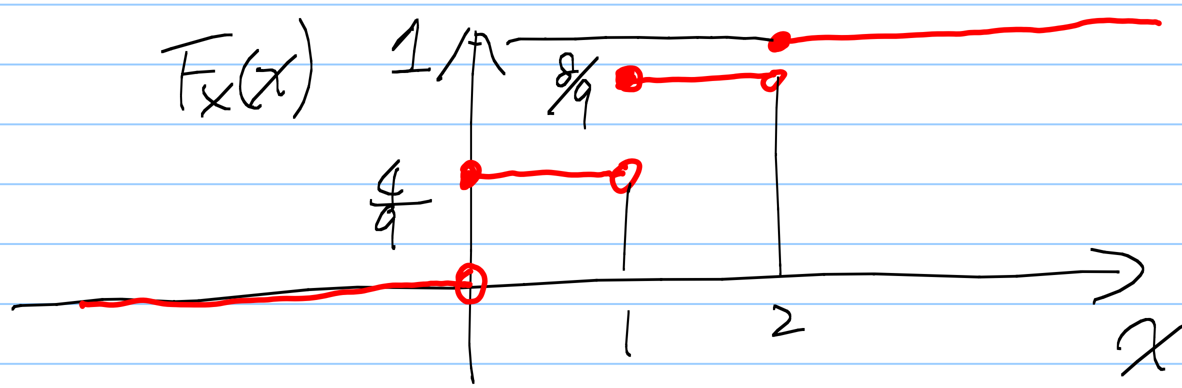


Ex: X is a binomial R.V with $n=2$, $p=\frac{1}{3}$

Find the cdf $F_X(x)$, Plot it.

Ans: $F_X(x) = P(X \le x)$

$$= \begin{cases} 0 & \text{if } x<0 \\ p_0 = \frac{4}{9} & 0 \le x < 1 \\ p_0+p_1 = \frac{8}{9} & 1 \le x < 2 \\ p_0+p_1+p_2 = 1 & 2 \le x \end{cases}$$



$\boxed{\text{II}}$ From cdf $F_X(x)$ back to p_k.

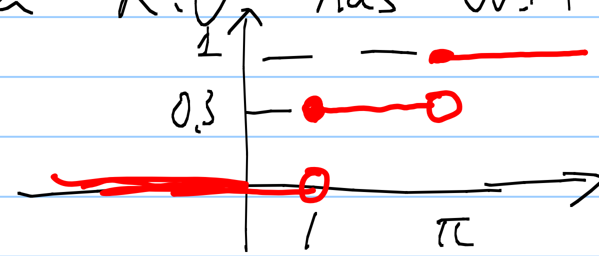
Ans: For each k value, p_k is the jump from $F_X(k-0.0001)$ to $F_X(k)$ (or sometimes we write it as $p_k = F_X(k) - F_X(k^-)$)

Ex: I can say a R.V. has W.A

① $S=\{1,\pi\}$

$P(X=1)=0.3$

$P(X=\pi)=0.7$



Or ② $F_X(x)=\begin{cases}0 & \text{if } x<1\\ 0.3 & \text{if } 1\le x<\pi\\ 0.3+0.7 & \text{if } \pi\le x\end{cases}$

Both ① and ② describe the same W.A.

As you can see, the pmf ① indeed corresponds to the jump in the cdf ②

* Summary:

① Given the pmf p_k, we can find the CDF $F_X(x)$ by counting.

② Given the CDF $F_X(x)$, we can find the pmf by location & the magnitude of the jumps.

Conti R.Vs:

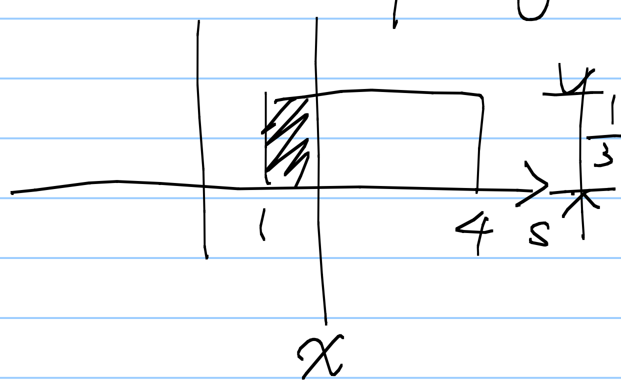
$\boxed{\text{III}}$ From pdf $f_X(x)$ to cdf $F_X(x)$

$$F_X(x) = P(X \le x)$$
$$= \int_{-\infty}^{x} f_X(s)\,ds$$

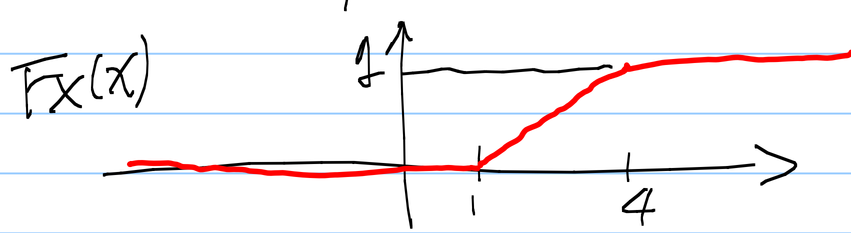
Ex: X is a uniform R.V over $(1, 4)$

What is the cdf of X.

Ans: $f_X(s) = \begin{cases} \frac{1}{4-1} & \text{if } 1 < s < 4 \\ 0 & \text{otherwise} \end{cases}$



$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \int_1^x \frac{1}{3}\,ds = \frac{x-1}{3} & \text{if } 1 \le x < 4 \\ \int_1^4 \frac{1}{3}\,ds = 1 & \text{if } 4 \le x \end{cases}$$



$\boxed{\text{IV}}$ How to start from a cdf $F_X(x)$ to derive the pdf $f_X(x)$?

Ans: $\because \ F_X(x) = \int_{-\infty}^{x} f_X(s)\,ds$

$\therefore \ f_X(x) = \frac{d}{dx} F_X(x)$ $\boxed{\text{Differentiation}}$

Exercise: Find & plot the cdf $F_X(x)$ for an exponential R.V.

Properties of a cdf $F_X(x)$

① $0 \le F_X(x) \le 1$.

② $\lim_{x \to \infty} F_X(x) = 1$

③ $\lim_{x \to -\infty} F_X(x) = 0$

④ $F_X(x)$ is non-decreasing

I.e. if $a < b$ then $F_X(a) \le F_X(b)$

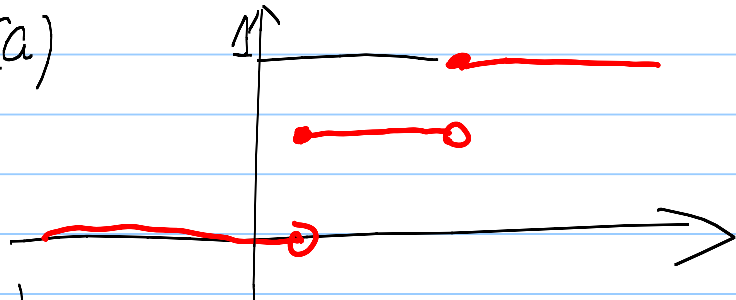
⑤ $F_X(x)$ is conti from the right.

Namely $F_X(x^-)$ may / may not be $F_X(x)$

but $F_X(x^+) = F_X(x)$

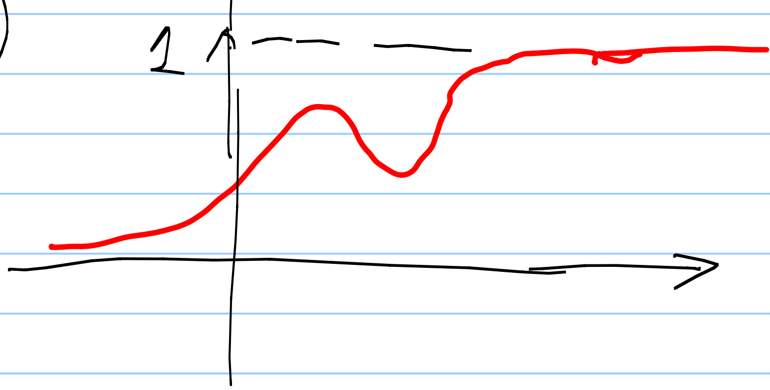
Ex: Which of the following figures can be a valid cdf $F_X(x)$

(a)

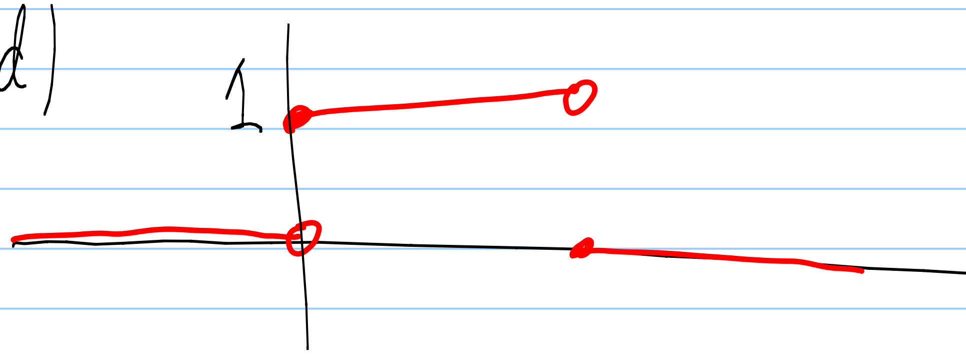


(b)

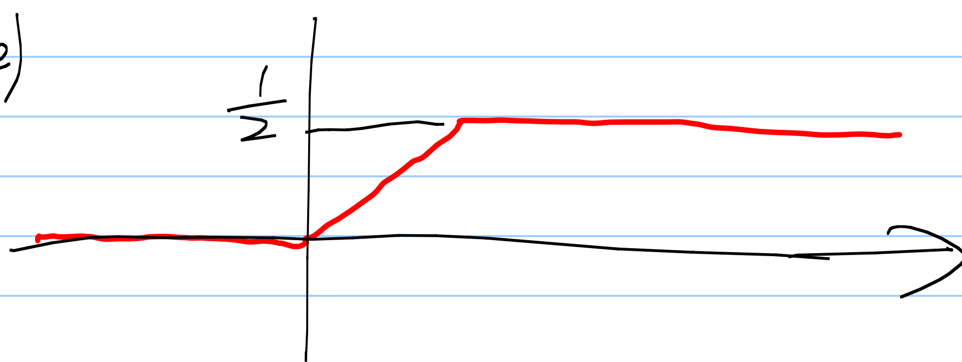
(c)



(d)



(e)



The $\boxed{\text{cumulative distribution function}}$ is less straightforward than the pmf & pdf ex: $F_X(x)$ the x can be fractional even when X is a discrete R.V.

However there are many advantages to use a cdf $F_X(x)$

* ① Applies to both discrete & conti R.V.

② Can be used to compute probabilities

Ex: Given a R.V. X with cdf $F_X(x)$

for example $F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$

(or $F_X(x) = U(x)$)

Find

Q: $P(X \le a) = ?$

Q: $P(a < X) = ?$

Q: $P(a < X \le b) = ?$

Q: $P(X < a) = ?$

Q: $P(a \le X) = ?$

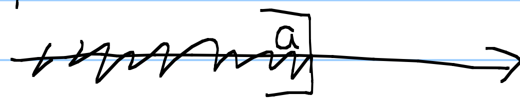
Q: $P(a \le X \le b) = ?$

in terms of $F_X(x)$

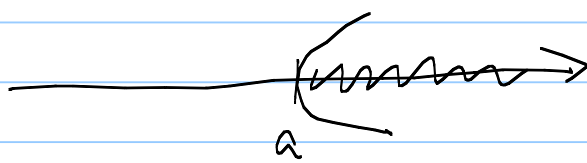
Q: $P(X=a) = ?$ Q: $P(a<X<b) = ?$

Q: $P(a \le X < b) = ?$

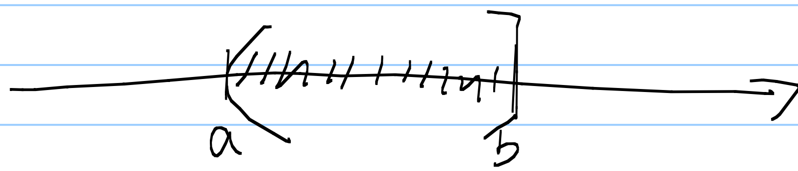
Ans: $P(X \le a) = F_X(a)$ by definition



$$P(X>a) = 1 - F_X(a)$$

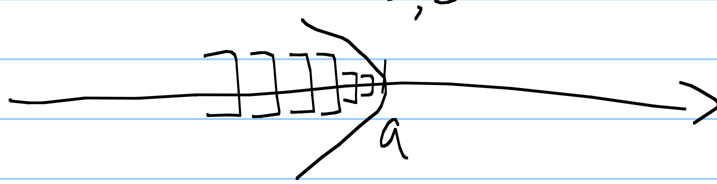


$$P(a < X \le b) = F_X(b) - F_X(a)$$

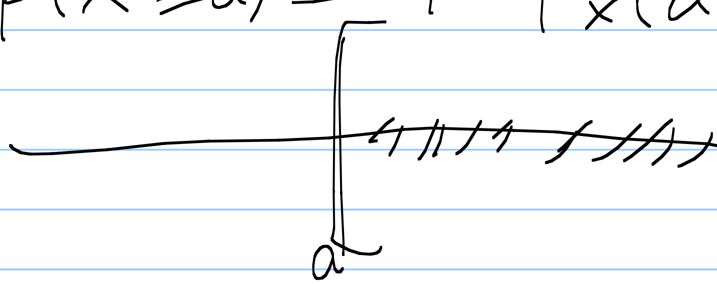


$$P(X<a) = \lim_{\varepsilon>0,\ \varepsilon \to 0} F_X(a-\varepsilon) \triangleq F_X(a^-)$$

Rewrite it as



$$P(X \ge a) = 1 - F_X(a^-)$$



$$P(a \le X \le b) = F_X(b) - F_X(a^-)$$

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$$P(X=a) = P(a \le X \le a) = F_X(a) - F_X(a^-)$$

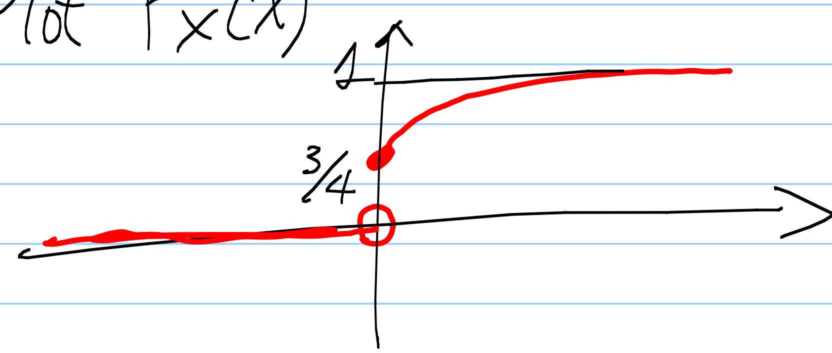
the jump

$$P(a<X<b) = F_X(b^-) - F_X(a)$$
$$P(a \le X<b) = F_X(b^-) - F_X(a^-)$$

Ex: HW6Q9 Prob 4.13

$$F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-\frac{1}{4}e^{-2x} & \text{if } 0 \le x \end{cases}$$

Plot $F_X(x)$



Q: $P(X \le 2)$, $P(X=0)$, $P(X<0)$,
$P(2<X<6)$, $P(X>10)$.

Ans: $F_X(2) = 1-\frac{1}{4}e^{-4}$

$F_X(0)-F_X(0^-) = (1-\frac{1}{4}e^{-0}) - 0 = \frac{3}{4}$

$F_X(0^-) = 0$

$F_X(6^-) - F_X(2) = \left(1-\frac{1}{4}e^{-12}\right) - \left(1-\frac{1}{4}e^{-4}\right)$

$= \frac{1}{4}(e^{-4} - e^{-12})$

$$1-F_X(10) = 1-\left(1-\frac{1}{4}e^{-20}\right)$$
$$= \frac{1}{4}e^{-20} \quad \#$$

Q $P(0 < X \leq 6) = ?$

Ans: $F_X(6) - F_X(0) = \left(1-\frac{1}{4}e^{-12}\right) - \left(\frac{3}{4}\right) = \frac{1}{4}(1-e^{-12})$

Advantage of using cdf:

(3) Characterizing R.V.s of mixed type.

Discrete R.V. $F_X(x)$ looks like Staircase. (containing only jumps)

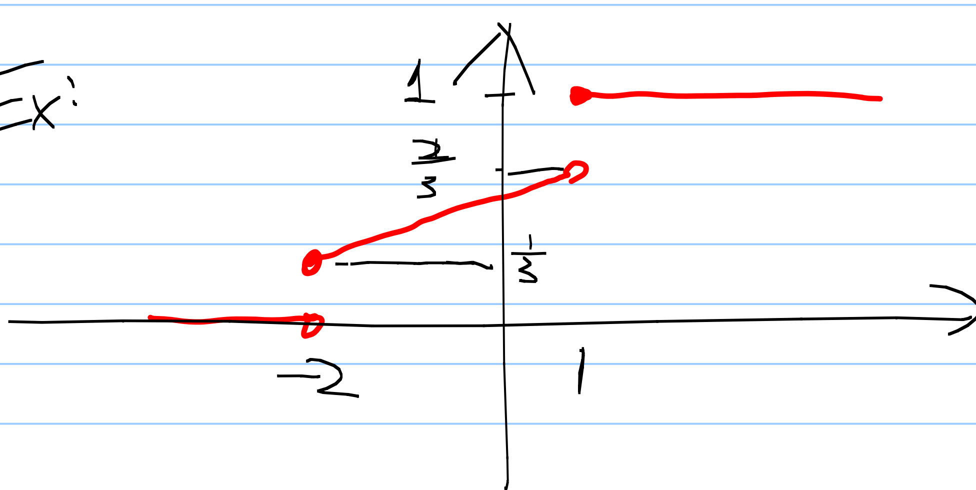
Conti $F_X(x)$ is continuously increasing.

R.V. of mixed type

$F_X(x)$ contains some jumps & some continuously rising regions.

Ex: HW6 Q9 Prob. 4.13.

Ex:



Ex: A real number X is chosen as follows: Flip a fair coin, if head, $X=\frac{1}{2}$

if tail: use a computer to pick randomly from $(0,1)$

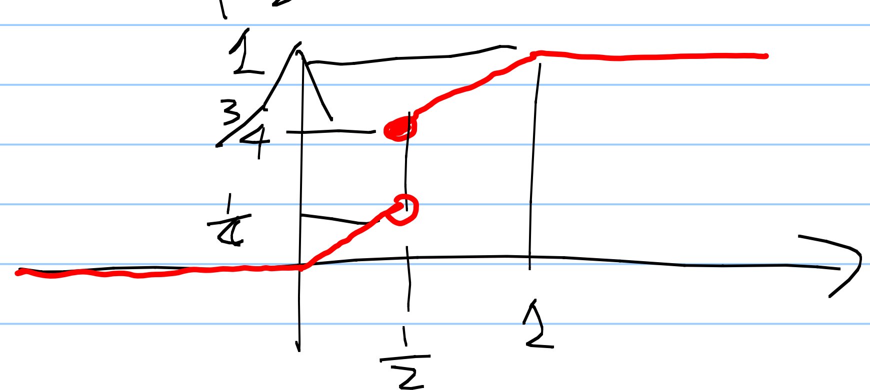
Q: $F_X(x)=?$

Ans: $F_X(x)=P(X\le x)$

$=P(X\le x \text{ and head})$

$+P(X\le x \text{ and tail})$

$$= \begin{cases} 0 & \text{if } x<0 \\ 0+\frac{1}{2}\int_0^x 1\,dx=\frac{1}{2}x & \text{if } 0\le x<\frac{1}{2} \\ \frac{1}{2}\times 1+\frac{1}{2}\int_0^x dx=\frac{1}{2}x+\frac{1}{2} & \text{if } \frac{1}{2}\le x<1 \\ \frac{1}{2}\times 1+\frac{1}{2}\times 1 & \text{if } 1\le x \end{cases}$$



Advantage

④ Find the pmf/pdf of a new R.V $Y = f(X)$

Similar to HW6 Q11, Q12

Ex: X is uniformly chosen from $[0, 2]$

$Y = X^2$

Find the pdf of X and Y.

Ans: pdf of X is straightforward

$$f_X(x) = \begin{cases} 1/2 & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

For pdf of Y, we use

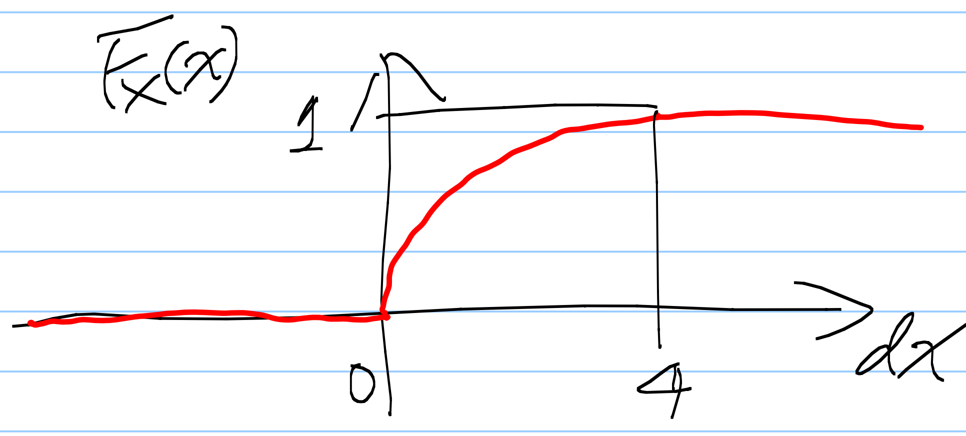
pdf of X $\Longrightarrow$ cdf of Y $\Longrightarrow$ pdf of Y.

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

Three cases:

$$F_Y(y) = \begin{cases} \boxed{0} & \text{if } \quad y<0 \\ \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \int_0^{\sqrt{y}} \frac{1}{2}\,dx = \boxed{\frac{\sqrt{y}}{2}} & 0\le y<4, \quad 0\le\sqrt{y}\le 2 \\ \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \int_0^{2} \frac{1}{2}\,dx = \boxed{1} & 4\le y, \ 2\le\sqrt{y} \end{cases}$$



$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0 & \text{if} \quad y<0 \\ \frac{1}{4} \times \frac{1}{\sqrt{y}} & \text{if } 0\le y<4 \\ 0 & 4\le y \end{cases}$$