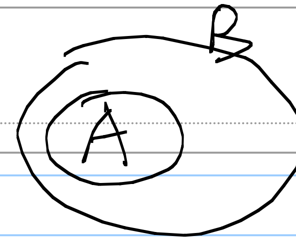
★ $\boxed{A \text{ implies } B} \equiv$ 

Ex: $A = \{\text{all multiples of } 4\}$. A implies B.
4, 8, 12
$B: \{\text{all multiples of } 2\}$. If X is a multiple of 4
2, 4, 6, 8, ...
then X must be a multiple of 2.

⑥ Commutativity

$A \cup B = B \cup A$ <u>OR : In at least one</u>

$A \cap B = B \cap A$ <u>AND : In ALL.</u>

⑦ Associativity


$A \cup (B \cup C) = (A \cup B) \cup C$

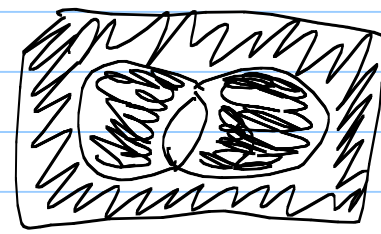

$A \cap (B \cap C) = (A \cap B) \cap C$

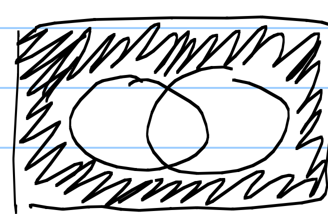
⑧ Distributivity


$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$


$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

⑨ Demorgan's Rule

$(A \cap B)^c = A^c \cup B^c$ 

$(A \cup B)^c = A^c \cap B^c$ 

Once we know how to include/exclude events/sets, we need to assign weights

A valid W.A satisfies the following 3 axioms

Axiom 1:

Each weight must be non-negative

$$P(A) \geq 0$$

Axiom 2: The total weight must be 1.

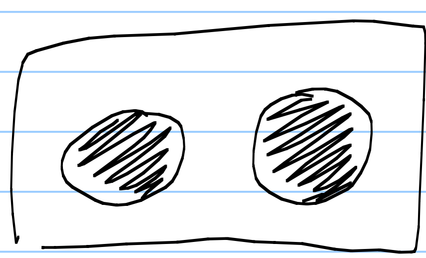
$$P(S) = 1.$$

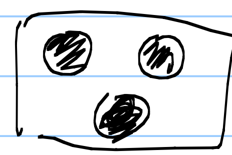
Axiom 3: If two <u>events</u> are disjoint, i.e, they can not happen simultaneouly.

$$\parallel A \cap B = \phi$$

then the weights of either A or B happens must be the sum of individual weights

$$P(A \cup B) = P(A) + P(B)$$





Similarly: if

$$A \cap B = B \cap C = C \cap A = \phi$$

Then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Axiom 3.1: If any two A_i, A_j are disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The above "axioms" are very intuitive and can be taken as <u>granted</u>. and used to show some non-intuitive results.

Corollary 1

$P(A^c) = 1 - P(A)$

Corollary 2

$P(A) \leq 1$

Corollary 3

$P(\phi) = 0$

Corollary 4

If $A_1 \cdots A_n$ are disjoint, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{k=1}^{n} P(A_k)$$

Corollary 5

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Ex: X is the outcome of a fair 6-faced die.

$$P(X \text{ is a prime or } X \geq 5)$$

$$= P(X = 2, 3, 5, 6) = \frac{4}{6}$$

$$= P(X = 2, 3, 5) + P(X = 5, 6) - P(X = 5)$$

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6}.$$

$\because$ The weight of $X = 5$

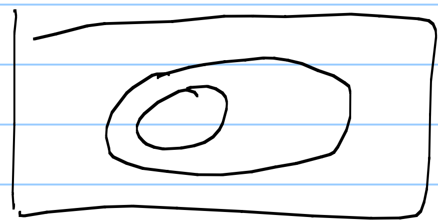
is double counted.

Corollary 7

If $A \subseteq B$, then $P(A) \leq P(B)$

A is a subset of B

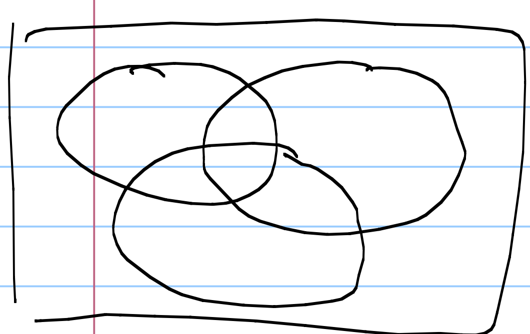
A implies B



Corollary 6

Question for the team: Explain the following "inclusion/exclusion" principle by the Venn Diagram

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$



$$- P(A \cap B) - P(B \cap C) - P(C \cap A)$$

$$+ P(A \cap B \cap C)$$

Q: $P(A \cup B \cup C \cup D) = ?$

019

① "We start by "set operations": how to include/exclude the events.

② Then discuss properties of a valid W.A.

③ The next question is how to construct a valid W.A by ourselves.

Case 1: The sample space is discrete. (ex: A card game, a coin)

Step 1: Specify the non-negative weight for each outcome. & Step 2: Make sure the total sum is 1.

Ex: A coin has two outcomes {H, T.}

We can assign $P(H) = \frac{1}{3}$ $P(T) = \frac{2}{3}$.

In many cases, we are interested in random experiments that have output being integers, then the weight assignment is described by

$$P_k = P(X = k)$$

This special type of experiments is called $\boxed{\text{"discrete random variable"}}$ & the associated weight assignment is called "discrete distribution"

* random: we do not know what the outcome will be.
Variable: The outcome is (usually) a number.
Discrete: Values are integers

* The p_k used for describing a discrete distribution (W.A) is called the $\boxed{\text{prob mass function (pmf)}}$

Example: A fair die is a discrete random variable & its distribution is described by pmf

$$p_1 = p_2 = p_3 = \cdots = p_6 = \frac{1}{6}, \quad \text{all other } p_k = 0$$

If we let 0 denote tail, 1 for head, then the previous coin experiment is a discrete R.V. & its distribution is described by the following pmf.

$$p_0 = \frac{2}{3} \quad p_1 = \frac{1}{3} \quad \text{all other } p_k = 0$$

Example: A discrete R.V has sample space $S = \{0, 1, 2, \cdots, \infty\}$ and it pmf (W.A) is

$$p_k = \frac{1}{4}\left(1 - \frac{1}{4}\right)^k \quad \text{for } k = 0, \cdots, \infty$$

Q: Is this a valid W.A

Ans: Check ① $p_k \geq 0$ for all k ✓

$\boxed{\text{Yes}}$ ② $\sum_{k=0}^{\infty} p_k = \frac{0.25}{1-(1-0.25)} = 1$ ✓

* We define W.A first & then make prob statements.
* Be careful when we try to design a W.A to "retro-fit" some prob. statement.

Ex: $S = \{1, 2, 3\}$ ex: 1 : sunny
2 : rainy
3 : snowy

If someone says that

The prob $(X \neq 2)$ is $5/8$

$\text{prob}(X \neq 1) = \frac{1}{4}$

Q: Are these two statements consistent? (Equivalently, can we find a valid W.A satisfying the above two statements?

A: Suppose we can, then we will have the pmf p_1, p_2, p_3

Then we have

$\boxed{\text{Not valid}}$

$$\begin{cases} p_1 + p_3 = 5/8 \\ p_2 + p_3 = \frac{1}{4} \\ p_1 + p_2 + p_3 = 1 \end{cases} \Longrightarrow p_1 = \frac{3}{4},\ p_2 = \frac{3}{8} \quad p_3 = -\frac{1}{8}$$

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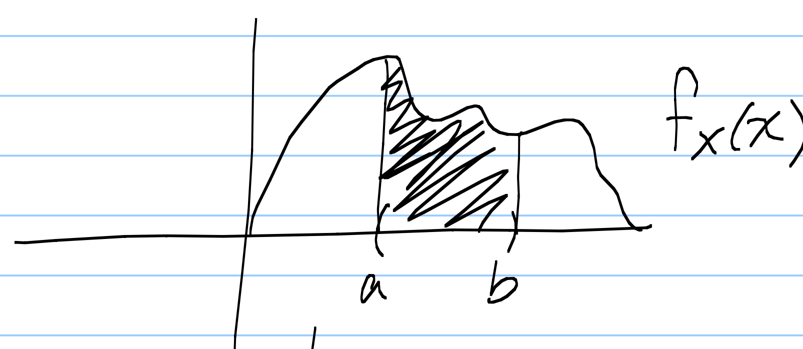
Case 2: Suppose the sample space is continuous, and the output of a random experiment is the real number. Ex: the temperature, the time that the instructor enters the classroom.

We say this type of random experiment is a $\boxed{\text{continuous random variable}}$, its W.A is a continuous distribution

The W.A is described by the area underneath a curve.

Namely



$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

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① To make sure the sum is 1, we need

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1.$$

② Note that $f_X(x)$ stays above zero ∵ all weights must be non-negative.

The curve $f_X(x)$ is termed the prob density function (pdf)

Ex: $f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Q: Is $f_X(x)$ a valid pdf (describing a valid W.A?

Ans: Check ① $f_X(x) \geq 0$ for all x ✓

② $\int_{-\infty}^{\infty} f_X(x)\,dx \overset{?}{=} 1$

$$= \int_0^{\infty} 0.5\, e^{-0.5x}\, dx = 1.$$

Since ①, ② $\Rightarrow$ Yes. $f_X(x)$ is valid.

Q. Do we need to have $f_X(x) \leq 1$ for all x?

Ans: No. in this example $f_X(0) = 2$.

1024

Example: Today's temperature is uniformly/equally likely distributed between (5 F, 40F)

What is the prob that $P(T>32)$?

Ans: Step 1: Find the sample space

$S=(5,40)$ | $S=\mathbb{R}$ any real number

Step 2: Construct the W.A. Since it is a continuous random variable. We need to specify a curve $f_X(x)$.

$\because$ Uniformly/equally likely, the curve should

be flat | be flat over (5.40)

$f_X(x)=c$ | $f_X(x)=\begin{cases} c & \text{if } 5<x<40 \\ 0 & 0 \end{cases}$

To be a valid W.A. |

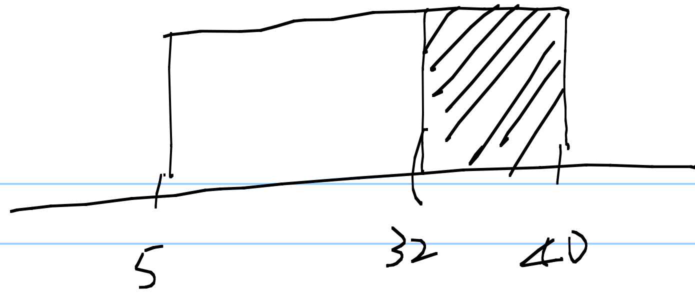
$\int_5^{40} f_X(x)dx=1$ | $\int_{-\infty}^{\infty} f_X(x)dx=1.$

$\Rightarrow 35c=1 \quad c=\frac{1}{35}$ | $=\int_5^{40} c\,dx=1$

$f_X(x)=\frac{1}{35}$ | $f_X(x)=\begin{cases} \frac{1}{35} & \text{if } 5<x<40 \\ 0 & \text{otherwise} \end{cases}$

Step 3: Count the prob of the desired event.

1025



$$P(T>32)=\int_{32}^{40}\frac{1}{35}\,dx = \frac{8}{35}$$

$$P(T>32)=\int_{32}^{\infty}f_X(x)\,dx = \int_{32}^{40}\frac{1}{35}\,dx = \frac{8}{35}$$

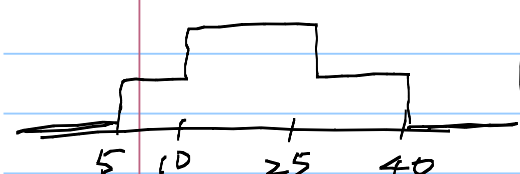
Example: Temperature is between 5 & 40. Suppose the prob that it is between (10, 25) is twice as likely is it is between (5, 10) or (25, 40)

Find $P(T>32)$

Ans: Step 1: the sample space is the same. $S=\mathbb{R}$

Step 2: The curve must be

$$f_X(x)=\begin{cases} 2c & \text{if } 10\le x<25 \\ c & \text{if } 5\le x<10 \text{ or } 25\le x<40 \\ 0 & \text{otherwise} \end{cases}$$



$\because \int_{-\infty}^{\infty} f_X(x) = 1$ we must have

$$\int_{5}^{10} c\,dx + \int_{10}^{25} 2c\,dx + \int_{25}^{40} c\,dx = 1$$

$\Rightarrow 50c = 1 \quad c = \frac{1}{50}$

$$f_X(x) = \begin{cases} \frac{1}{25} & \text{if } 10 \le x < 25 \\ \frac{1}{50} & \text{if } 5 \le x < 10,\ 25 \le x < 40 \\ 0 & \text{otherwise} \end{cases}$$

Step 3: Count the weight

$$P(T > 32) = \int_{32}^{\infty} f_X(x)\,dx$$

$$= \int_{32}^{40} \frac{1}{50}\,dx = \frac{8}{50}$$

$= 16\%$

Note Title 1/19/2011

HW2 Q13:

Q For any valid W.A. show that

$$P(-\infty < X \leq r) \leq P(-\infty < X \leq s)$$

if $r \leq s$

Ans: Since $r \leq s$

$$P(-\infty < X \leq s) = P(-\infty < X \leq r) + \underbrace{P(r < X \leq s)}_{\geq 0}$$

Since

$$\Rightarrow P(-\infty < X \leq s) \geq P(-\infty < X \leq r)$$

Q: Suppose

$$P(-\infty < X \leq r) = p_r$$

$$P(-\infty < X \leq s) = p_s$$

what is $P(r < X \leq s)$?

Ans: $\because$ $p_s = p_r + P(r < X \leq s)$

$\therefore$ $P(r < X \leq s) = p_s - p_r$

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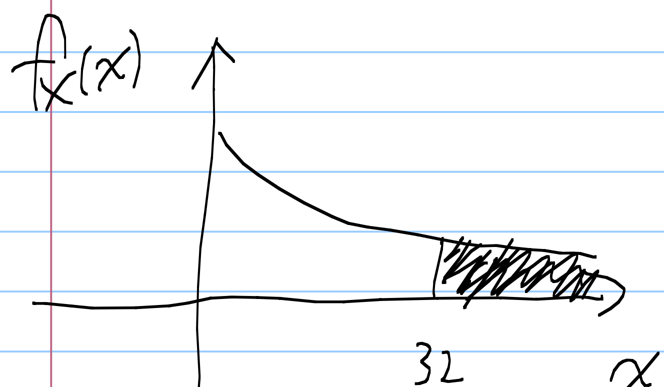
Example: A continuous R.V. has sample space $S=[0,\infty)$ (all positive real numbers)

and the prob density function is

$$f_X(x)=\begin{cases}3e^{-3x} & \text{if } x \geq 0\\ 0 & \text{otherwise}\end{cases}$$

Q: What is the prob $X > 32$

Ans:



$$\int_{32}^{\infty} f_X(x)\,dx$$

$$= \int_{32}^{\infty} 3e^{-3x}\,dx$$

$$= e^{-96}$$ #

* Conditional prob
(Or the "Relative Frequency")

Example: Two teams play a game.
& the weather can be sunny/rainy.

Q: Sample space?
A: $S = \{$(A win, sunny), (A win, rainy)
(B win, sunny), (B win, rainy)$\}$

Q: Construct a valid W.A?
Ans: $\because$ S is discrete
$\therefore$ By pmf. We use a table to represent the sample space

	S	R
A win	$\frac{1}{4}$	$\frac{1}{4}$
B win	$\frac{1}{6}$	$\frac{1}{3}$

Q: P(A wins) = ? Ans: $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Q: What is the prob ~~that~~ A wins <u>conditioned</u> on that it rains?

The conditional prob should be
P(A win | Rainy) as the

$$\frac{\text{\# of times A win \& Rainy}}{\text{\# of times Rainy}} = \frac{P(\text{A wins \& rains})}{P(\text{rains})}$$