PURDUE

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EXAM TITLE

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1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
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5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

Midterm #2 of ECE302, Section 3

8–9pm, Thursday, October 12, 2023, SMTH 108.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam may contain some multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Please do not write on the back of each page. Any work on the back of the pages will not be scanned and thus will not be graded.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

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Question 1: [21%, Work-out question] Consider a random variable X with the corresponding pdf being

$$f(x) = \begin{cases} \frac{2x}{3} & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Answer the following questions.

1. [12%] Define the two events $A = \{|X| \leq 1.5\}$ and $B = \left\{\sqrt{\frac{3}{2}} \leq X \leq \sqrt{\frac{33}{10}}\right\}$, respectively. Are events A and B independent? This is not a yes/no question. An answer without any justification will receive zero point.

2. [3%] What is the value of the *first central moment* of X?

3. [6%] What is the value of the *third moment* of X?

   Hint: If you do not know the answer to Q1.3, you can answer the following alternative question. What is the value of $E(X)$? You will receive 4 points for Q1.3 if your answer is correct.

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Question 2: [14%, Work-out question] Suppose random variable X is of Bernoulli distribution with $p = \frac{2}{3}$. Answer the following questions.

1. [7%] Find the value of $E(1 + \pi X^2 - \sqrt{X})$.
2. [7%] Find the closed-form expression of the function $\Phi(s) = E(e^{sX})$.

   Hint: If you do not know how to find the expression of Q2.2, you can answer the following alternative question instead. What is the value of $\Phi(0) = E(e^{0X})$, i.e., the value of $\Phi(s)$ when plugging in $s = 0$. You will receive 4 points for Q2.2 if your answer is correct.

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Question 3: [15%, Work-out question] Consider a binomial random variable X with parameter $n = 4$ and $p = \frac{1}{3}$. Answer the following questions:

1. [4%] Find the value of $P(X = 2)$. Your answer must be of the form $\frac{a}{b}$ where a and b are two integers. For example, you may write $P(X = 2) = \frac{15}{37}$. Your answer must not be something like $\binom{6}{2} \cdot \frac{4}{55}$.

   The hint in the end of this question may be useful when computing the answer of Q3.1.

2. [11%] Let $i = 1$ and $j = 3$. Prove the following inequality

$$P(X \geq i + j | X \geq j) \neq P(X \geq i) \tag{1}$$

Hint: we know that $0! = 1$; $1! = 1$; $2! = 2$; $3! = 6$; and $4! = 24$.
Also $2^1 = 2$; $2^2 = 4$; $2^3 = 8$; and $2^4 = 16$.
Finally, $3^1 = 3$; $3^2 = 9$; $3^3 = 27$ and $3^4 = 81$.

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Question 4: [16%, Work-out question] Suppose the average incoming traffic flow of a web-page server is "5400 requests per hour". Please use either an exponential distribution or a Poisson distribution when answering the following questions.

1. [8%] What is the probability that "the arrival time of the *first* web-page request is > 10 seconds"?

   Hint: Your answer can be of a form similar to $\frac{3^{12}}{5!}\pi^5 + \frac{2^5}{3!}e^5$. There is no need to further simplify it.

2. [8%] We know that if there are ≥ 20 requests in a 5-second interval, the server will crash. What is the *conditional probability* that "there are strictly larger than 15 requests in a 5-second interval, given that the server has not crashed in that 5-second interval"?

   Hint: Your answer will be a fractional number. The numerator can be of a form similar to $\frac{3^{12}}{5!}\pi^5 + \frac{2^5}{3!}e^5$. There is no need to further simplify it. The denominator can be of a form similar to $\sum_{k=3}^{\infty} \frac{3^k}{(k+5)!}$. There is no need to further simplify the summation.

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Question 5: [10%, Work-out question] We know that random variable X is a Gaussian random variable with parameters $\mu = \pi$ and $\sigma^2 = 10$.

Find out the value of $E\left(5(X+3)^2\right)$.

Hint 1: If you do not know how to solve this question, you can find the values of $E(X)$ and $\mathsf{Var}(X)$ instead. You will receive 4 points if your answers of both $E(X)$ and $\mathsf{Var}(X)$ are correct.

Hint 2: $\mathsf{Var}(X) = E\left(X^2\right) - \left(E(X)\right)^2$.

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Question 6: [24%, Work-out question] Consider a function $f(x)$ that converts any given real-valued number x into either 1, or 2 in the following way.

$$f(x) = \begin{cases} 1 & \text{if } x < 4 \\ 2 & \text{if } 4 \le x \end{cases} \tag{2}$$

For example, $f(0.99) = 1$, $f(\pi) = 1$, and $f(100.35) = 2$.

We use a computer to generate a random number X that is *exponentially distributed* with $\lambda = 0.5$. (If you look up the random variable table, it also means that $E(X) = \frac{1}{\lambda} = 2$ and $\mathsf{Var}(X) = \frac{1}{\lambda^2} = 4$.)

We then generate $Y = f(X)$. Namely, the randomly generated X is fed into the function to create a new value Y. Obviously, because X is random, so is Y. For example, if the randomly generated $X = 0.99$, then $Y = f(X) = f(0.99) = 1$. If the randomly generated $X = 100.35$, then $Y = f(X) = f(100.35) = 2$. That is, different X value will lead to different Y value.

Answer the following questions.

1. [4%] What is the probability $P(Y = 1)$? Hint: it is equivalent to asking what is the probability $P(f(X) = 1)$.
2. [4%] What is the probability $P(Y = 2)$? Hint: it is equivalent to asking what is the probability $P(f(X) = 2)$.
3. [3%] Find the value of the expectation $E(f(X))$. Hint: You can either start from scratch or you can use the answers of the previous two sub-questions.
4. [1.5%] What does the acronym pmf stand for?
5. [1.5%] What does the acronym cdf stand for?
6. [10%] Plot the cdf $F_Y(y)$ of the random variable Y for the range of $-1 \le y \le 6$. Please carefully mark your cdf plot with solid/empty circles to represent the cdf in a correct way.

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Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \leq p \leq 1$.

  $E(X) = p$, $\text{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\text{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\text{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.