ECE 302-003 Homework #9 Solution

Fall 2023

Question 96:

$X \sim N(m, \sigma)$

$Y = aX + b$

$E[Y] = am + b = m'$

$$E[Y^2] = a^2 E[X^2] + 2ab E[X] + b^2$$
$$= a^2 (Var(X) + (E[X])^2) + 2ab E[X] + b^2$$
$$= a^2(\sigma^2 + m^2) + 2abm + b^2$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = a^2\sigma^2 + a^2m^2 + 2abm + b^2 - (a^2m^2 + 2abm + b^2)$$
$$= a^2\sigma^2$$

$m' = am + b$

$\sigma' = \sqrt{a^2\sigma^2}$

$\sigma' = a\sigma$

$a = \frac{\sigma'}{\sigma}$

$b = m' - \frac{\sigma'}{\sigma} m$

Question 97:

a: $c = 100 \quad b = 150 \qquad X \sim Uniform(0, b)$

$$P(X>c) = 1 - \int_0^{100} \frac{1}{150}\, dx = 1 - \frac{100}{150} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P(X>c) \le \frac{E[X]}{c} = \frac{75}{100} = \frac{3}{4}$$

Markov Inequality $- P(X>c) = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$

b: $c = 100 \quad b = 150 \quad \lambda = 2 \qquad X \sim Exponential(\lambda)$

$$P(X>c) = \int_{100}^{\infty} 2e^{-2x}\, dx = -e^{-2x}\Big|_{100}^{\infty} = e^{-200}$$

$$P(X>c) \le \frac{E[X]}{c}$$

$$\le \frac{\frac{1}{2}}{100} = \frac{1}{200}$$

$$E[X] = \int_0^{\infty} x\, 2e^{-2x} dx\, dx \qquad u = x \quad dv = 2e^{-2x}dx$$

$$du = dx \quad v = -e^{-2x}$$

$$= -xe^{-2x}\Big|_0^{\infty} + \frac{1}{2}\int_0^{\infty} 2e^{-2x} dx$$

$$= 0 + \frac{1}{2}$$

Markov Inequality $- P(X>c) \approx 0.005$

c: $c = 100 \quad b = 150 \qquad X \sim Uniform(-b, b)$

$\cancel{P(X>c) = \int_{100}^{150} \frac{1}{300} dx = \frac{50}{300} = \frac{1}{6}}$

$\cancel{P(X>c) \le E[X]}$

$$P(|X| \ge c) = \int_{100}^{150} \frac{1}{300}\, dx + \int_{-150}^{-100} \frac{1}{300}\, dx = 2\,\frac{50}{300} = \frac{1}{3}$$

$$P(|X| \ge c) \le \frac{Var(X)}{c^2}$$

$$E[X^2] = \int_{-b}^{b} x^2 \frac{1}{300} dx = \frac{1}{300}\frac{1}{3}x^3\Big|_{-b}^{b} = \frac{1}{900}(b^3 + b^3)$$

$$= \frac{2(150)^3}{900} = 7500$$

$$\le \frac{7500}{10000} = \frac{3}{4}$$

Chebyshev Inequality $- P(|X| \ge c) = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$

Question 98:

$X \sim \text{Binomial}(p, n)$

$E[X] = np \qquad Var(X) = np(1-p)$

$Y = \frac{1}{n} X$

$E[Y] = \frac{1}{n} E[X] = p$

~~Var~~ $E[Y^2] = \frac{1}{n^2} E[X^2] = \frac{1}{n^2}\left(Var(X) + (E[X])^2\right) = \frac{1}{n^2}\left(np(1-p) + n^2p^2\right)$

$Var(Y) = E[Y^2] - (E[Y])^2$

$= \frac{p(1-p)}{n} + p^2 - p^2 = \frac{p(1-p)}{n}$

$P(|Y-p| > a) \leq \frac{Var(Y)}{a^2}$ by the Chebyshev Inequality

$\leq \frac{p(1-p)}{a^2 n}$

$\lim_{n\to\infty} P(|Y-p| > a) = \lim_{n\to\infty} \frac{p(1-p)}{a^2} \frac{1}{n} = 0$

Question 99:

$X = \max(W_1, W_2) \qquad Y = \min(W_1, W_2)$

a: $S = \{$ all 2^4 possible 4-bit strings (combinations of 0 & 1)$\}$

$S_{XY} = \{0,1,2\} \times \{0,1,2\}$

0000 →	(0,0)	1000 →	(1,0)
0001	($\cancel{1}$,0)	1001	(1,1)
0010	($\cancel{1}$,0)	1010	(1,1)
0011	(2,0)	1011	(2,1)
0100	(1,0)	1100	(2,0)
0101	(1,1)	1101	(2,1)
0110	(1,1)	1110	(2,1)
0111	(2,1)	1111	(2,2)

b: $P((0,0)) = \frac{1}{16}$ $\qquad P((1,0)) = \frac{4}{16}$ $\qquad P((2,0)) = \frac{2}{16}$

$P((0,1)) = 0$ $\qquad P((1,1)) = \frac{4}{16}$ $\qquad P((2,1)) = \frac{4}{16}$

$P((0,2)) = 0$ $\qquad P((1,2)) = 0$ $\qquad P((2,2)) = \frac{1}{16}$

c: $P(X=Y) = P((0,0)) + P((1,1)) + P((2,2)) = \frac{6}{16}$

d: $P((0,0)) = P(0000) = (\frac{1}{2})(\frac{1}{2})(\frac{1}{4})(\frac{1}{4}) = \frac{1}{64}$

$P((0,1)) = 0$

$P((0,2)) = 0$

$P((1,0)) = (\frac{1}{4})(\frac{1}{4})(\frac{1}{2})(\frac{1}{2}) + (\frac{1}{4})(\frac{1}{4})(\frac{1}{2})(\frac{1}{2}) + (\frac{1}{4})(\frac{3}{4})(\frac{1}{2})(\frac{1}{2}) + (\frac{3}{4})(\frac{1}{4})(\frac{1}{2})(\frac{1}{2})$

$= \frac{1}{2}\left[\frac{1}{16} + \frac{3}{16}\right] = \frac{4}{32} = \frac{8}{64}$

$P((1,1)) = 4(\frac{1}{4})(\frac{3}{4})(\frac{1}{4}) = \frac{12}{64}$

$P((1,2)) = 0$

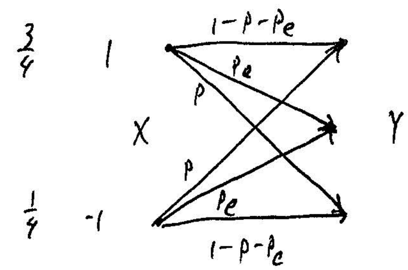
$$P((2,0)) = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) = \frac{10}{64}$$

$$P((2,1)) = 2\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) + 2\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) = \frac{6+18}{64} = \frac{24}{64}$$

$$P((2,2)) = \left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) = \frac{9}{64}$$

$$P(X=Y) = \frac{1+12+9}{64} = \frac{22}{64}$$

Question 100:



a: $S = \{-1, 1\}$

$S_{XY} = \{(1,-1), (1,0), (1,1), (-1,-1), (-1,0), (-1,1)\}$

b: $P((1,-1)) = \frac{3}{4}p$

$P((1,0)) = \frac{3}{4}p_e$

$P((1,1)) = \frac{3}{4}(1-p-p_e)$

$P((-1,-1)) = \frac{1}{4}(1-p-p_e)$

$P((-1,0)) = \frac{1}{4}p_e$

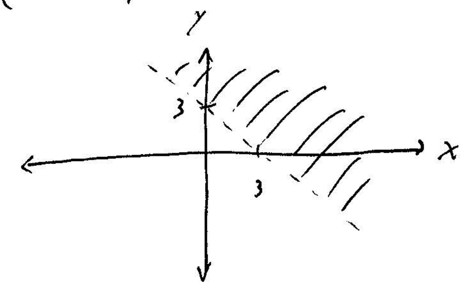
$P((-1,1)) = \frac{1}{4}p$

c: $P(X \neq Y) = \frac{3}{4}p + \frac{3}{4}p_e + \frac{1}{4}p_e + \frac{1}{4}p = p + p_e$

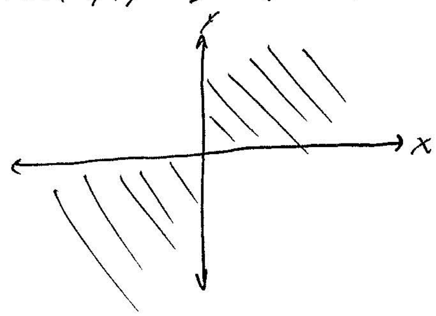
$P(Y=0) = \frac{3}{4}p_e + \frac{1}{4}p_e = p_e$

Question 101:

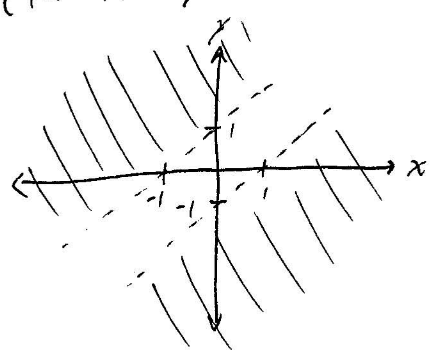
a: $\{X + Y > 3\}$



b: $\{\min(X, Y) > 0\} \cup \{\max(X, Y) < 0\}$



d: $\{|X - Y| \geq 1\}$

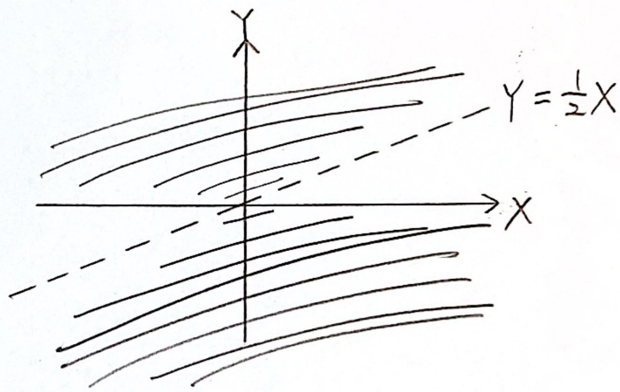


Question 102:

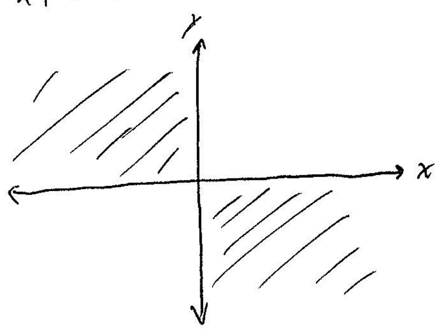
$f: \{X/Y < 2\}$

1° If $Y>0$, $Y > \frac{1}{2}X$

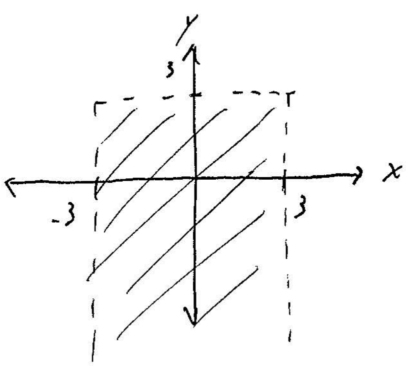
2° If $Y<0$, $Y < \frac{1}{2}X$



$h: \{XY<0\}$



$i: \{\max(|X|, Y) < 3\}$



Question 103:

$$P_X(0) = P((0,0)) + P((0,1)) + P((0,2)) = \frac{1}{16}$$

$$P_X(1) = \frac{8}{16}$$

$$P_X(2) = \frac{7}{16}$$

$$P_Y(0) = P((0,0)) + P((1,0)) + P((2,0)) = \frac{7}{16}$$

$$P_Y(1) = \frac{8}{16}$$

$$P_Y(2) = \frac{1}{16}$$

Question 104:

a: (i) $p_X(-1) = \frac{1}{6} + \frac{1}{6} + 0 = \frac{1}{3}$

$p_X(0) = \frac{1}{3}$

$p_X(1) = \frac{1}{3}$

$p_Y(-1) = \frac{1}{3}$

$p_Y(0) = \frac{1}{3}$

$p_Y(1) = \frac{1}{3}$

(ii) $p_X(-1) = \frac{1}{3}$

$p_X(0) = \frac{1}{3}$

$p_X(1) = \frac{1}{3}$

$p_Y(-1) = \frac{1}{3}$

$p_Y(0) = \frac{1}{3}$

$p_Y(1) = \frac{1}{3}$

(iii) $p_X(-1) = \frac{1}{3}$

$p_X(0) = \frac{1}{3}$

$p_X(1) = \frac{1}{3}$

$p_Y(-1) = \frac{1}{3}$

$p_Y(0) = \frac{1}{3}$

$p_Y(1) = \frac{1}{3}$

b: (i) $P(X>0) = \frac{1}{3}$ $\quad P(X \geq Y) = 3(\frac{1}{6}) = \frac{1}{2}$ $\quad P(X = -Y) = \frac{1}{6}$

(ii) $P(X>0) = \frac{1}{3}$ $\quad P(X \geq Y) = 6(\frac{1}{9}) = \frac{2}{3}$ $\quad P(X = -Y) = 3(\frac{1}{9}) = \frac{1}{3}$

(iii) $P(X>0) = \frac{1}{3}$ $\quad P(X \geq Y) = 1$ $\quad P(X = -Y) = \frac{1}{3}$

Question 105:

$X = r\cos(2\pi \Theta \tfrac{1}{8})$ $\quad Y = r\sin(2\pi \Theta \tfrac{1}{8})$ $\quad \Theta \sim \text{Uniform}(0, 1, \dots, 7)$

a: $S_\Theta = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$$S_{XY} = \{ \cancel{r\cos} (r,0), (r\cos(\tfrac{\pi}{4}), r\sin(\tfrac{\pi}{4})), (0,r), (r\cos(\tfrac{3\pi}{4}), r\sin(\tfrac{3\pi}{4})), (-r,0), (r\cos(\tfrac{5\pi}{4}), r\sin(\tfrac{5\pi}{4})), (0,-r), (r\cos(\tfrac{7\pi}{8}), r\sin(\tfrac{7\pi}{8}))\}$$

$$= \{(r,0), (\tfrac{r}{\sqrt{2}}, \tfrac{r}{\sqrt{2}}), (0,r), (-\tfrac{r}{\sqrt{2}}, \tfrac{r}{\sqrt{2}}), (-r,0), (-\tfrac{r}{\sqrt{2}}, \tfrac{-r}{\sqrt{2}}), (0,-r), (\tfrac{r}{\sqrt{2}}, -\tfrac{r}{\sqrt{2}})\}$$

b: $P((r,0)) = P((\tfrac{r}{\sqrt{2}}, \tfrac{r}{\sqrt{2}})) = P((0,r)) = P((-\tfrac{r}{\sqrt{2}}, \tfrac{r}{\sqrt{2}})) = P((-r,0)) = P((-\tfrac{r}{\sqrt{2}}, \tfrac{-r}{\sqrt{2}})) = P((0,-r)) = P((\tfrac{r}{\sqrt{2}}, \tfrac{-r}{\sqrt{2}})) = \tfrac{1}{8}$

c: $P_X(-r) = \tfrac{1}{8}$ $\quad P_Y(-r) = \tfrac{1}{8}$

$P_X(-\tfrac{r}{\sqrt{2}}) = \tfrac{2}{8}$ $\quad P_Y(-\tfrac{r}{\sqrt{2}}) = \tfrac{2}{8}$

$P_X(0) = \tfrac{2}{8}$ $\quad P_Y(0) = \tfrac{2}{8}$

$P_X(\tfrac{r}{\sqrt{2}}) = \tfrac{2}{8}$ $\quad P_Y(\tfrac{r}{\sqrt{2}}) = \tfrac{2}{8}$

$P_X(r) = \tfrac{1}{8}$ $\quad P_Y(r) = \tfrac{1}{8}$

d: $P(X=0) = \tfrac{2}{8}$

$P(Y \le \tfrac{r}{\sqrt{2}}) = \tfrac{7}{8}$

$P(X \ge \tfrac{r}{\sqrt{2}}, Y \ge \tfrac{r}{\sqrt{2}}) = \tfrac{1}{8}$

$P(X < \tfrac{-r}{\sqrt{2}}) = \tfrac{1}{8}$