ECE 302-003, Homework #9
Due date: Wednesday 11/15/2023, 11:59pm;

`https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html`

Question 96: [Basic] Problem 4.85.

4.85. The exam grades in a certain class have a Gaussian pdf with mean m and standard deviation σ. Find the constants a and b so that the random variable $y = aX + b$ has a Gaussian pdf with mean m' and standard deviation σ'.

Question 97: [Basic] Problem 4.97(a) with $c = 100$ and $b = 150$. Problem 4.97(b) with $c = 100$, $b = 150$, and $\lambda = 2$. Problem 4.99(a) with $c = 100$ and $b = 150$.

4.97. Compare the Markov inequality and the exact probability for the event $\{X > c\}$ as a function of c for:
(a) X is a uniform random variable in the interval $[0, b]$.
(b) X is an exponential random variable with parameter λ.
(c) X is a Pareto random variable with $\alpha > 1$.
(d) X is a Rayleigh random variable.

4.99. Compare the Chebyshev inequality and the exact probability for the event $\{|X - m| > c\}$ as a function of c for:
(a) X is a uniform random variable in the interval $[-b, b]$.
(b) X is a Laplacian random variable with parameter α.
(c) X is a zero-mean Gaussian random variable.
(d) X is a binomial random variable with $n = 10$, $p = 0.5$; $n = 50$, $p = 0.5$.

Question 98: [Intermediate/Exam Level] Problem 4.100.

4.100. Let X be the number of successes in n Bernoulli trials where the probability of success is p. Let $Y = X/n$ be the average number of successes per trial. Apply the Chebyshev inequality to the event $\{|Y - p| > a\}$. What happens as $n \to \infty$?

Question 99: [Basic] Problem 5.1. (Hint: The most difficult part of this question is how the question is described. Basically, Carlos and Michael each flips a coin twice and there are totally four random outcomes since totally 4 coins have been flipped and each can be head or tail. Focusing on the first two outcomes, Carlos computes the number of heads and denotes it by W_1. Focusing on the last two outcomes, Michael computes the number of heads and denotes it by W_2. Then $X = \max(W_1, W_2)$ and $Y = \min(W_1, W_2)$.)

5.1. Let X be the maximum and let Y be the minimum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the probabilities for all values of (X, Y).

(c) Find $P[X = Y]$.

(d) Repeat parts b and c if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

Question 100: [Basic] Problem 5.3

5.3. The input X to a communication channel is "-1"or "1", with respective probabilities 1/4 and 3/4. The output of the channel Y is equal to: the corresponding input X with probability $1 - p - p_e$; $-X$ with probability p; 0 with probability p_e.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the probabilities for all values of (X, Y).

(c) Find $P[X \neq Y]$, $P[Y = 0]$.

Question 101: [Basic] Problem 5.8(a,c,d).

Question 102: [Basic] Problem 5.8(f,h,i).

5.8. For the pair of random variables (X, Y) sketch the region of the plane corresponding to the following events. Identify which events are of product form.

(a) $\{X + Y > 3\}$.

(b) $\{e^X > Ye^3\}$.

(c) $\{\min(X, Y) > 0\} \cup \{\max\{X, Y) < 0\}$.

(d) $\{|X - Y| \geq 1\}$.

(e) $\{|X/Y| > 2\}$.

(f) $\{X/Y < 2\}$.

(g) $\{X^3 > Y\}$.

(h) $\{XY < 0\}$.

(i) $\{\max(|X|, Y) < 3\}$.

Question 103: [Basic] Problem 5.9(b)

5.9. **(a)** Find and sketch $p_{X,Y}(x, y)$ in Problem 5.1 when using a fair coin.
(b) Find $p_X(x)$ and $p_Y(y)$.
(c) Repeat parts a and b if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

Question 104: [Intermediate/Exam Level] Problem 5.11.

5.11. **(a)** Find the marginal pmf's for the pairs of random variables with the indicated joint pmf.

(i)

X/Y	−1	0	1
−1	1/6	1/6	0
0	0	0	1/3
1	1/6	1/6	0

(ii)

X/Y	−1	0	1
−1	1/9	1/9	1/9
0	1/9	1/9	1/9
1	1/9	1/9	1/9

(iii)

X/Y	−1	0	1
−1	1/3	0	0
0	0	1/3	0
1	0	0	1/3

(b) Find the probability of the events $A = \{X > 0\}$, $B = \{X \geq Y\}$, and $C = \{X = -Y\}$ for the above joint pmf's.

Question 105: [Intermediate/Exam Level] Problem 5.12. In (a), please change the statement to "write down the original sample space S_Θ and the new sample space S_{XY}.

5.12. A modem transmits a two-dimensional signal (X, Y) given by:

$$X = r\cos(2\pi\Theta/8) \quad \text{and} \quad Y = r\sin(2\pi\Theta/8)$$

where Θ is a discrete uniform random variable in the set $\{0, 1, 2, \ldots, 7\}$.
(a) Show the mapping from S to S_{XY}, the range of the pair (X, Y).
(b) Find the joint pmf of X and Y.
(c) Find the marginal pmf of X and of Y.
(d) Find the probability of the following events: $A = \{X = 0\}$, $B = \{Y \leq r/\sqrt{2}\}$, $C = \{X \geq r/\sqrt{2}, Y \geq r/\sqrt{2}\}$, $D = \{X < -r/\sqrt{2}\}$.