ECE 302-003 Homework #8 Solution

Fall 2023

Question 85:

$$Y = 2X + 3.75 \qquad E[X] = 1.2 \qquad Var(X) = 2.8$$

$$E[Y] = E[2X + 3.75] = 2E[X] + 3.75 = 2(1.2) + 3.75 = 6.15$$

$$E[Y^2] = E[(2X+3.75)^2] = E[4X^2 + 15X + 14.0625]$$

$$= 4E[X^2] + 15E[X] + 14.0625$$

$$= 4(Var(X) + (E[X])^2) + 15E[X] + 14.0625$$

$$= 4(2.8 + (1.2)^2) + 15(1.2) + 14.0625 = 49.0225$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = 49.0225 - (6.15)^2 = 11.2$$

Question 86:

Let X be the number of heads out of n trials.

$$P_X(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

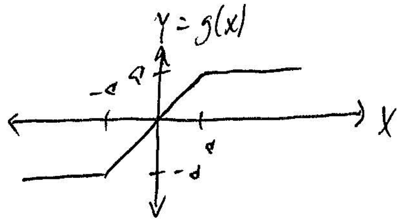
$$E(X) = np \quad , \quad Var(X) = np(1-p)$$

Also, we have $Y = X - (n-X) = 2X - n$.

$$\therefore E(Y) = E(2X-n) = 2E(X) - n = 2np - n = n(2p-1) \#$$

$$Var(Y) = 4Var(X) = 4np(1-p) \#$$

Question 87:



c: $a = \frac{1}{2}$ $\quad f_X(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0 & ow \end{cases}$

$$\int_{-1}^{1} c(1-x^2)dx = 1 \Rightarrow c\left[x - \tfrac{1}{3}x^3\right]_{-1}^{1} = c\left((1-\tfrac{1}{3}) - (-1+\tfrac{1}{3})\right) = c\,\tfrac{4}{3} \Rightarrow c = \tfrac{3}{4}$$

$$P(Y = -\tfrac{1}{2}) = P(X \le -\tfrac{1}{2}) = \int_{-1}^{-\frac{1}{2}} \tfrac{3}{4}(1-x^2)dx = \tfrac{3}{4}\left[x - \tfrac{1}{3}x^3\right]_{-1}^{-\frac{1}{2}}$$

$$= \tfrac{3}{4}\left[(-\tfrac{1}{2} + \tfrac{1}{24}) - (-1 + \tfrac{1}{3})\right] = \tfrac{3}{4}\left(\tfrac{2}{3} - \tfrac{1}{2} + \tfrac{1}{24}\right) = \tfrac{3}{4}\left(\tfrac{16-12+1}{24}\right) = \tfrac{5}{32}$$

$$P(Y = 1/2) = P(X \ge \tfrac{1}{2}) = \int_{\frac{1}{2}}^{1} \tfrac{3}{4}(1-x^2)dx = \tfrac{5}{32}$$

$$f_Y(y) = \tfrac{3}{4}(1-y^2)\left[\mathcal{U}(y+\tfrac{1}{2}) - \mathcal{U}(y-\tfrac{1}{2})\right] + \tfrac{5}{32}\left(\delta(y+\tfrac{1}{2}) + \delta(y-\tfrac{1}{2})\right)$$

For $-\frac{1}{2} < y < \frac{1}{2}$:

$$F_Y(y) = \tfrac{5}{32} + \int_{-\frac{1}{2}}^{y} \tfrac{3}{4}(1-s^2)ds = \tfrac{5}{32} + \tfrac{3}{4}\left[s - \tfrac{1}{3}s^3\right]_{-\frac{1}{2}}^{y}$$

$$= \tfrac{5}{32} + \tfrac{3}{4}\left[(y - \tfrac{1}{3}y^3) - (-\tfrac{1}{2} + \tfrac{1}{24})\right] = \tfrac{3}{4}y - \tfrac{1}{4}y^3 + \tfrac{48-4+15}{96}$$

$$= \tfrac{3}{4}y - \tfrac{1}{4}y^3 + \tfrac{59}{96}$$

$$F_Y(y) = \begin{cases} 0 & y < -\frac{1}{2} \\ \frac{3}{4}y - \frac{1}{4}y^3 + \frac{59}{96} & -\frac{1}{2} \le y < \frac{1}{2} \\ 1 & y \ge \frac{1}{2} \end{cases}$$

Question 88:

$Y = |X|$

a: $F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y)$

$= F_X(y) - F_X(-y)$

$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(y) - f_X(-y)(-1) = \begin{cases} f_X(y) + f_X(-y) & y \ge 0 \\ 0 & y < 0 \end{cases}$

b: $P(y < Y \le y + dy) = P(y < |X| \le y + dy)$

$= P(y < X \le y + dy) + P(-y - dy \le X < -y)$

$= \begin{cases} f_X(y) + f_X(-y) & y \ge 0 \\ 0 & y < 0 \end{cases}$ Yes

c: $f_X(-x) = f_X(x) \Rightarrow f_Y(y) = \begin{cases} 2f_X(y) & y \ge 0 \\ 0 & y < 0 \end{cases}$

Question 89:

$X \sim Exp(0.3) \qquad \alpha = 1,\ b = 3.5$

a: $F_X(x) = \int_0^x 0.3 e^{-0.3s}\, ds = -e^{-0.3s}\Big|_0^x = 1 - e^{-0.3x}$

$$P(\alpha \le X \le b) = F_X(b) - F_X(\alpha) = \left(1 - e^{-0.3(3.5)}\right) - \left(1 - e^{-0.3(1)}\right)$$

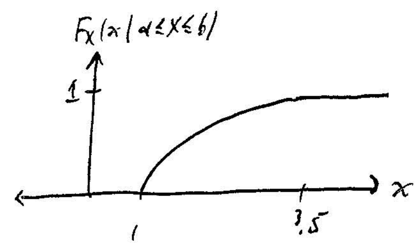
$$= e^{-0.3} - e^{-1.05}$$

$$f(x \mid \alpha \le X \le b) = \begin{cases} \dfrac{f_X(x)}{P(\alpha \le X \le b)} & \alpha \le x \le b \\ 0 & ow \end{cases}$$

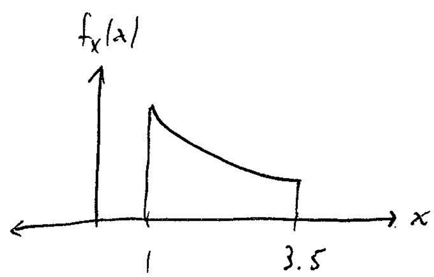
$$F_X(x \mid \alpha \le X \le b) = P(X \le x \mid \alpha \le X \le b) = \frac{1}{P(\alpha \le X \le b)} \int_\alpha^x f_X(s)\, ds$$

$$= \frac{1}{e^{-0.3} - e^{-1.05}} \int_1^x 0.3 e^{-0.3s}\, ds = \frac{1}{e^{-0.3} - e^{-1.05}} \left(e^{-0.3} - e^{-0.3x}\right)$$

$$F_X(x \mid \alpha \le X \le b) = \begin{cases} 0 & x < 1 \\ \dfrac{e^{-0.3} - e^{-0.3x}}{e^{-0.3} - e^{-1.05}} & 1 \le x \le 3.5 \\ 1 & x > 3.5 \end{cases}$$



b: $$f_X(x \mid \alpha \le X \le b) = \begin{cases} \dfrac{0.3 e^{-0.3x}}{e^{-0.3} - e^{-1.05}} & 1 \le x \le 3.5 \\ 0 & ow \end{cases}$$



Question 90:

$X \sim \text{Poisson}(3.4) \qquad a = 0.2 \qquad b = 4.2$

$P(X=k) = \frac{e^{-3.4}}{k!}(3.4)^k \quad \text{for } k = 0, 1, 2, \ldots$

$a: P(0.2 \le X \le 4.2) = P(1 \le X \le 4) = e^{-3.4}\left[\frac{3.4}{1} + \frac{(3.4)^2}{2} + \frac{(3.4)^3}{6} + \frac{(3.4)^4}{24}\right] = 0.7108$

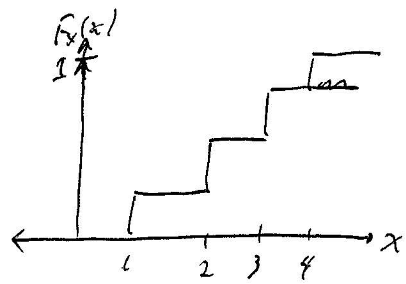
$F_X(x) = P(X \le x)$

$F_X(x \mid a \le X \le b) = P(X \le x \mid a \le X \le b)$

$$P(X=k \mid a \le X \le b) = \frac{P(X=k)}{P(a \le X \le b)} = \begin{cases} \dfrac{\frac{e^{-3.4}}{k!}(3.4)^k}{0.7108} & k = 1, 2, 3, 4 \\ 0 & \text{ow} \end{cases}$$

$$F_X(x \mid a \le X \le b) = \begin{cases} 0 & x < 1 \\ \dfrac{3.4e^{-3.4}}{0.7108} & 1 \le x < 2 \\ \dfrac{3.4e^{-3.4} + e^{-3.4}\left(\frac{(3.4)^2}{2}\right)}{0.7108} & 2 \le x < 3 \\ 3.4e^{-3.4} + e^{-3.4}\left(\frac{(3.4)^2}{2}\right) + e^{-3.4}\left(\frac{(3.4)^3}{6}\right) & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

$$= \begin{cases} 0 & x < 1 \\ 0.1596 & 1 \le x < 2 \\ 0.4310 & 2 \le x < 3 \\ 0.7386 & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$



$$b: P(X=k \mid 0.2 \le X \le 4.2) = \begin{cases} \dfrac{\frac{e^{-3.4}}{k!}(3.4)^k}{0.7108} & k = 1, 2, 3, 4 \\ 0 & \text{ow} \end{cases}$$

Question 91:

$S_X = \{1, 2, 3, 4\} \qquad X < 4$

a: $P_k = \frac{P_1}{k}$ for all $k \in S_X$

$\sum_{k=1}^{3} P_k = 1 = P_1 + \frac{1}{2}P_1 + \frac{1}{3}P_1 = P_1\left(\frac{6+3+2}{6}\right) = P_1\left(\frac{11}{6}\right)$

$P_1 = \frac{6}{11} \qquad P_2 = \frac{3}{11} \qquad P_3 = \frac{2}{11}$

b: $P_{k+1} = \frac{P_k}{2}$

$\sum_{k=1}^{3} P_k = 1 = P_1 + \frac{1}{2}P_1 + \frac{1}{4}P_1 = P_1\left(\frac{4+2+1}{4}\right) = P_1\left(\frac{7}{4}\right)$

$P_1 = \frac{4}{7} \qquad P_2 = \frac{2}{7} \qquad P_3 = \frac{1}{7}$

c: $P_{k+1} = \frac{P_k}{2^k}$

$\sum_{k=1}^{3} P_k = 1 = P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3 = P_1\left(\frac{7}{4}\right)$

$P_1 = \frac{4}{7} \qquad P_2 = \frac{2}{7} \qquad P_3 = \frac{1}{7}$

Question 92:

$X \sim U(-b, b)$

a: $\Phi_X(x) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$

$= \int_{-b}^{b} \frac{1}{2b} e^{j\omega x} dx = \frac{1}{2b} \frac{1}{j\omega} e^{j\omega x} \Big|_{-b}^{b} = \frac{1}{b\omega} \frac{1}{2j} \left(e^{j\omega b} - e^{-j\omega b} \right)$

$= \frac{\sin(\omega b)}{\omega b}$

b: $E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_X(\omega) \Big|_{\omega=0}$

$E[X] = \frac{1}{j} \frac{d}{d\omega} \Phi_X(\omega) \Big|_{\omega=0} = \frac{1}{j}$

$= \frac{1}{j}\left[\frac{\cos(\omega b)}{\omega} + \frac{\sin(\omega b)}{b}\left(-\frac{1}{\omega^2}\right)\right]\Big|_{\omega=0} = \frac{1}{j}\left[\frac{\cos(\omega b)\omega b - \sin(\omega b)}{\omega^2 b}\right]\Big|_{\omega=0}$

$= \frac{1}{j}\left[\frac{-b^2\omega \sin(\omega b) + b\cos(\omega b) - b\cos(\omega b)}{2\omega b}\right]\Big|_{\omega=0} = \frac{1}{j}\left[\frac{-b^2\omega \sin(\omega b)}{2\omega b}\right]\Big|_{\omega=0}$

$= \frac{1}{j}\left[\frac{-b\sin(\omega b)}{2}\right]\Big|_{\omega=0} = 0$

$E[X^2] = \frac{1}{j^2}\frac{d^2}{d\omega^2}\Phi_X(\omega)\Big|_{\omega=0} = \frac{1}{j}\frac{d}{d\omega}\left[\frac{1}{j}\frac{\cos(\omega b)\omega b - \sin(\omega b)}{\omega^2 b}\right]\Big|_{\omega=0}$

$= \frac{1}{j}\left(\frac{1}{j}\right)\left[\frac{b\cos(\omega b) + \omega b^2(-\sin(\omega b)) - b\cos(\omega b)}{\omega^2 b} + \frac{-2}{\omega^3 b}\left(\cos(\omega b)\omega b - \sin(\omega b)\right)\right]\Big|_{\omega=0}$

$= (-1)\left[\frac{-b\sin(\omega b)}{\omega} + \frac{-2\omega b\cos(\omega b) + 2\sin(\omega b)}{\omega^3 b}\right]\Big|_{\omega=0}$

$= (-1)\left[\frac{-b^2\cos(\omega b)}{(1)} + \frac{-2b\cos(\omega b) + 2\omega b^2\sin(\omega b) + 2b\cos(\omega b)}{3\omega^2 b}\right]\Big|_{\omega=0}$

$= (-1)\left(-b^2 + \frac{2b^2\sin(\omega b) + 2\omega b^3\cos(\omega b)}{6\omega b}\right)\Big|_{\omega=0}$

$= (-1)\left(-b^2 + \frac{+2b^3\cos(\omega b) + 2b^3\cos(\omega b) - 2\omega b^4\sin(\omega b)}{6b}\right)\Big|_{\omega=0} = b^2 - \frac{2}{3}b^2 = \frac{1}{3}b^2$

$Var(X) = E[X^2] - (E[X])^2 = \frac{1}{3}b^2$

Question 93:

$$f_X(x) = \frac{1}{2}\alpha e^{-\alpha|x|}$$

a: $$\Phi_X(\omega) = \int_{-\infty}^{\infty} \frac{1}{2}\alpha e^{-\alpha|x|} e^{j\omega x} dx$$

$$= \frac{\alpha}{2}\int_{-\infty}^{0} e^{\alpha x} e^{j\omega x} dx + \frac{\alpha}{2}\int_{0}^{\infty} e^{-\alpha x} e^{j\omega x} dx$$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{x(\alpha+j\omega)} dx + \int_{0}^{\infty} e^{-x(\alpha-j\omega)} dx\right]$$

$$= \frac{\alpha}{2}\left[\frac{1}{\alpha+j\omega} e^{x(\alpha+j\omega)}\Big|_{-\infty}^{0} + \frac{-1}{\alpha-j\omega} e^{-x(\alpha-j\omega)}\Big|_{0}^{\infty}\right] = \frac{\alpha}{2}\left[\frac{1}{\alpha+j\omega} + \frac{1}{\alpha-j\omega}\right]$$

$$= \frac{\alpha^2}{(\alpha^2+\omega^2)}$$

(b) $$E(X) = \frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega)\Big|_{\omega=0} = \frac{1}{j}\frac{d}{d\omega}\frac{\alpha^2}{(\alpha^2+\omega^2)}\Big|_{\omega=0}$$

$$= \frac{1}{j}\left[-\frac{\alpha^2}{(\alpha^2+\omega^2)^2}\cdot 2\omega\right]_{\omega=0} = -\frac{2\alpha^2}{j}\frac{\omega}{(\alpha^2+\omega^2)^2}\Big|_{\omega=0}$$

$$= 0 \quad \#$$

$$E(X^2) = \frac{1}{j^2}\frac{d^2}{d\omega^2}\Phi_X(\omega)\Big|_{\omega=0} = (-1)\frac{d}{d\omega}\left(-\frac{2\alpha^2\omega}{(\alpha^2+\omega^2)^2}\right)\Big|_{\omega=0}$$

$$= \left[2\alpha^2\frac{1}{(\alpha^2+\omega^2)^2} + (2\alpha^2\omega)(-2)\frac{1}{(\alpha^2+\omega^2)^3}\cdot 2\omega\right]_{\omega=0}$$

$$= \left[\frac{2\alpha^2}{\alpha^4} + 0\right] = \frac{2}{\alpha^2}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{\alpha^2} - 0 = \frac{2}{\alpha^2} \quad \#$$

Question 94:

$$Y = aX + b \qquad X \sim N(\mu, \sigma^2)$$

$$\Phi_Y(\omega) = E[e^{j\omega Y}] = E[e^{j\omega(aX+b)}] = e^{j\omega b} E[e^{j\omega a X}]$$

$$= e^{j\omega b} \Phi_X(a\omega)$$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{j\omega x} e^{-\frac{(x^2 - 2\mu x + \mu^2)}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2 - x(2\mu - 2j\omega\sigma^2) + \mu^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x - (\mu - j\omega\sigma^2))^2} e^{-\frac{1}{2\sigma^2}(2j\omega\mu\sigma^2 + \omega^2\sigma^4)} dx$$

$$= e^{-j\omega\mu} e^{-\frac{1}{2}\omega^2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-(\mu - j\omega\sigma^2))^2}{2\sigma^2}} dx}_{1}$$

$$= e^{-j\omega\mu} e^{-\frac{1}{2}\omega^2\sigma^2}$$

$$\Phi_Y(\omega) = e^{j\omega b}\left[e^{-ja\omega\mu} e^{-\frac{1}{2}a^2\omega^2\sigma^2}\right]$$

Question 95:

a: $P(X=k) = pq^k$ for $k = 0, 1, 2, \ldots$

$$G_X(z) = E[z^N] = \sum_{k=0}^{\infty} P_X(k)\, z^k$$

$$= \sum_{k=0}^{\infty} p(1-p)^k z^k = p \sum_{k=0}^{\infty} ((1-p)z)^k = \frac{p}{1-((1-p)z)} \quad \text{for } |(1-p)z| < 1$$

b: $$E[X] = \frac{d}{dz} G_X(z) \Big|_{z=1}$$

$$= p \frac{-1}{(1-(1-p)z)^2} (-(1-p)) \Big|_{z=1} = \frac{p(1-p)}{(1-(1-p)z)^2} \Big|_{z=1} = \frac{p(1-p)}{(1-(1-p))^2} = \frac{(1-p)}{p}$$

$$E[X^2] = E[X(X-1)] + E[X]$$

$$= \frac{d^2}{dz} G_X(z) \Big|_{z=1} + \frac{(1-p)}{p}$$

$$= \frac{d}{dz} \frac{p(1-p)}{(1-(1-p)z)^2} \Big|_{z=1} + \frac{(1-p)}{p}$$

$$= \frac{2p(1-p)(1-(1-p)z)(1-p)}{(1-(1-p)z)^4} \Big|_{z=1} + \frac{(1-p)}{p}$$

$$= \frac{2p(1-p)^2}{(1-(1-p)z)^3} \Big|_{z=1} + \frac{(1-p)}{p}$$

$$= \frac{2(1-p)^2}{p^2} + \frac{(1-p)}{p}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2(1-p)^2}{p^2} + \frac{1-p}{p} - \left(\frac{1-p}{p}\right)^2$$

$$= \frac{(1-p)^2}{p^2} + \frac{p(1-p)}{p^2} = \frac{1-p}{p^2} \quad \#$$