**ECE 302-003, Homework #8**
**Due date: Wednesday 11/08/2023, 11:59pm;**

https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html

*Question 85:* [Basic] Consider two random variables $X$ and $Y$, and suppose $Y = 2X + 3.75$. Suppose we also know the mean of $X$ is 1.2 and the variance of $X$ is 2.8. Find out $E(Y)$, $E(Y^2)$, and $\text{Var}(Y)$. Hint: $E(Y) = E(2X + 3.75)$ and $E(Y^2) = E(4X^2 + 15X + 3.75^2)$.

*Question 86:* [Basic]

A coin is tossed $n$ times. Let the random variable $Y$ be the difference between the number of heads and the number of tails. Assume $P(\text{head}) = p$. (This is a question similar to Problem 3.9.)

Compute $E(Y)$ and $\text{Var}(Y)$.

(Hint: Relate $Y$ to a binomial random variable $X$ with parameters $n$, $p$. More specifically, $Y$ can be written as a linear function of $X$. Basically, we are solving this question in a similar way as in Q85.)

*Question 87:* [Intermediate/Exam Level] Problem 4.82. Only do the subquestion corresponding to Problem 4.54(c).

**4.82.** Find the cdf and pdf of the output of the limiter in Problem 4.54 parts b, c, and d.

**4.17.** A random variable $X$ has pdf:

$$f_X(x) = \begin{cases} c(1 - x^2) & -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

**(a)** Find $c$ and plot the pdf.
**(b)** Plot the cdf of $X$.
**(c)** Find $P[X = 0]$, $P[0 < X < 0.5]$, and $P[|X - 0.5| < 0.25]$.

**4.54.** A limiter is shown in Fig. P4.2.

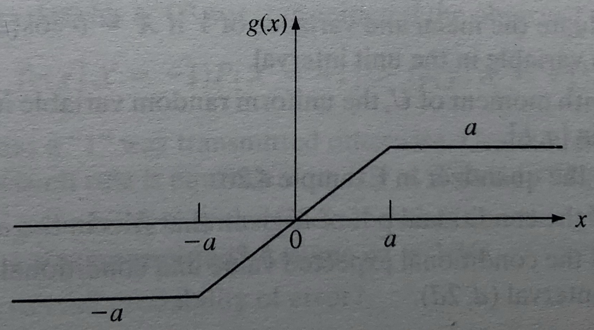


FIGURE P4.2

**(a)** Find an expression for the mean and variance of $Y = g(X)$ for an arbitrary continuous random variable $X$.

**(b)** Evaluate the mean and variance if $X$ is a Laplacian random variable with $\lambda = a = 1$.

**(c)** Repeat part (b) if $X$ is from Problem 4.17 with $a = 1/2$.

**(d)** Evaluate the mean and variance if $X = U^3$ where $U$ is a uniform random variable in the unit interval, $[-1, 1]$ and $a = 1/2$.

*Question 88:* [Intermediate/Exam Level] Problem 4.88.

**4.88.** Let $Y = |X|$ be the output of a full-wave rectifier with input voltage $X$.

**(a)** Find the cdf of $Y$ by finding the equivalent event of $\{Y \le y\}$. Find the pdf of $Y$ by differentiation of the cdf.

**(b)** Find the pdf of $Y$ by finding the equivalent event of $\{y < Y \le y + dy\}$. Does the answer agree with part a?

**(c)** What is the pdf of $Y$ if the $f_X(x)$ is an even function of $x$?

*Question 89:* [Basic] Assume $X$ is an exponential random variable with parameter $\lambda = 0.3$ and also assume $a = 1$, $b = 3.5$. Complete Problem 4.35.

**4.35.** **(a)** Find and plot $F_X(x \mid a \le X \le b)$. Compare $F_X(x \mid a \le X \le b)$ to $F_X(x)$.

**(b)** Find and plot $f_X(x \mid a \le X \le b)$.

*Question 90:* [Basic] Assume $X$ is a Poisson random variable with parameter $\lambda = 3.4$ and also assume $a = 0.2$, $b = 4.2$. Complete Problem 4.35(a) and find the conditional pmf $P(X = k|a \le X \le b)$.

*Question 91:* [Basic] Problem 3.36.

**3.36.** Find the conditional pmf for the quaternary information source in Problem 3.12, parts a, b, and c given that $X < 4$.

**3.12.** Consider an information source that produces binary pairs that we designate as $S_X = \{1, 2, 3, 4\}$. Find and plot the pmf in the following cases:

**(a)** $p_k = p_1/k$ for all $k$ in $S_X$.

**(b)** $p_{k+1} = p_k/2$ for $k = 2, 3, 4$.

**(c)** $p_{k+1} = p_k/2^k$ for $k = 2, 3, 4$.

**(d)** Can the random variables in parts a, b, and c be extended to take on values in the set $\{1, 2, \ldots\}$? If yes, specify the pmf of the resulting random variables. If no, explain why not.

*Question 92:* [Basic] Problem 4.102.

**4.102. (a)** Find the characteristic function of the uniform random variable in $[-b, b]$.

**(b)** Find the mean and variance of $X$ by applying the moment theorem.

*Question 93:* [Basic] Problem 4.103. Do not be scared by the "Laplacian random variable," which is simply a random variable with sample space being $\mathbf{R}$ and pdf $f_X(x) = 0.5\alpha e^{-\alpha|x|}$for a positive parameter $\alpha > 0$.

**4.103. (a)** Find the characteristic function of the Laplacian random variable.

**(b)** Find the mean and variance of $X$ by applying the moment theorem.

*Question 94:* [Intermediate/Exam Level] Problem 4.106.

**4.106.** Find the characteristic function of $Y = aX + b$ where $X$ is a Gaussian random variable. *Hint:* Use Eq. (4.79).

### 4.7.1 The Characteristic Function

The **characteristic function** of a random variable $X$ is defined by

$$\Phi_X(\omega) = E[e^{j\omega X}] \tag{4.79a}$$

$$= \int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\, dx, \tag{4.79b}$$

where $j = \sqrt{-1}$ is the imaginary unit number. The two expressions on the right-hand side motivate two interpretations of the characteristic function. In the first expression, $\Phi_X(\omega)$ can be viewed as the expected value of a function of $X$, $e^{j\omega X}$, in which the parameter $\omega$ is left unspecified. In the second expression, $\Phi_X(\omega)$ is simply the Fourier transform of the pdf $f_X(x)$ (with a reversal in the sign of the exponent). Both of these interpretations prove useful in different contexts.

*Question 95:* [Basic] Problem 4.109.

**4.109.** **(a)** Find the probability generating function of the geometric random variable.
**(b)** Find the mean and variance of the geometric random variable from its pgf.