ECE 302-003 Homework #7 Solution

Fall 2023

Question 71:

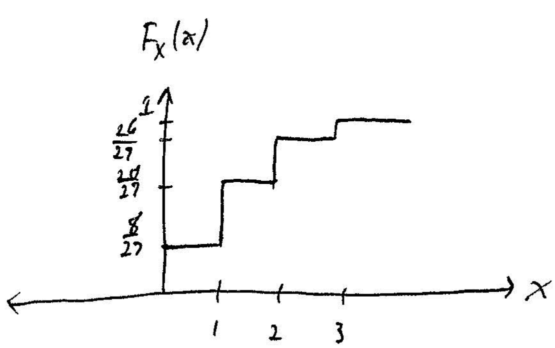
$$P(X=k) = \binom{3}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{3-k}$$

$$P(X=0) = (1)(1)\left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$P(X=1) = (3)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

$$P(X=2) = (3)\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{2}{9}$$

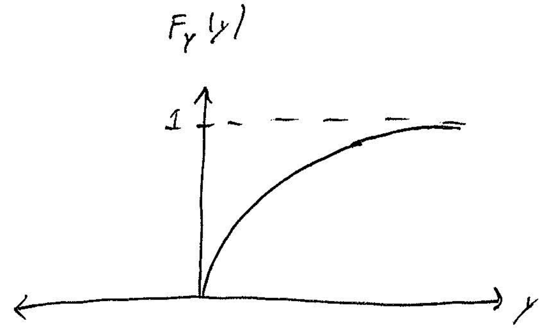
$$P(X=3) = (1)\left(\frac{1}{3}\right)^3(1) = \frac{1}{27}$$



$$f_Y(y) = \frac{4}{3}e^{-\frac{4}{3}y}$$

$$P(Y\le y) = F_Y(y) = \int_0^y \frac{4}{3}e^{-\frac{4}{3}s}\,ds$$

$$= -e^{-\frac{4}{3}s}\Big|_0^y = 1 - e^{-\frac{4}{3}y}$$



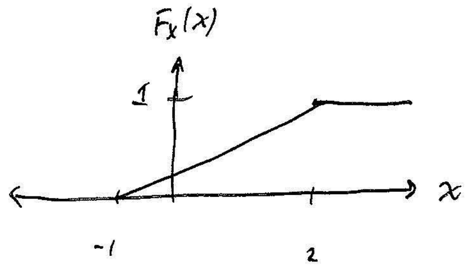
Question 72:

$X \sim \text{Uniform}(-1, 2)$

a: $f_X(x) = \begin{cases} \frac{1}{3} & -1 \le x \le 2 \\ 0 & \end{cases}$

$F_X(x) = P(X \le x) = \int_{-1}^{x} \frac{1}{3}\,ds = \frac{1}{3} s \Big|_{-1}^{x} = \frac{1}{3}(x+1) \quad \text{for} \quad -1 \le x \le 2$

$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3}(x+1) & -1 \le x \le 2 \\ 1 & x > 2 \end{cases}$



b: $P(X \le 0) = F_X(0) = \frac{1}{3}$

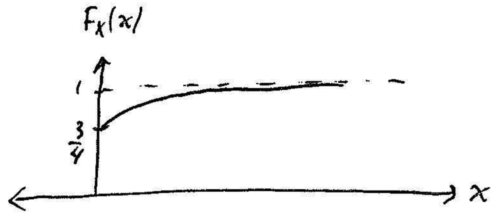
$P(|X - \frac{1}{2}| < 1) = P(-\frac{1}{2} < X < \frac{3}{2}) = F_X(\frac{3}{2}) - F_X(-\frac{1}{2}) = \frac{1}{3}(\frac{5}{2}) - \frac{1}{3}(\frac{1}{2}) = \frac{2}{3}$

$P(X > -\frac{1}{2}) = 1 - P(X \le -\frac{1}{2}) = 1 - F_X(-\frac{1}{2}) = 1 - \frac{1}{3}(\frac{1}{2}) = \frac{5}{6}$

Question 73:

$$F_X(x) = \begin{cases} 0 & x<0 \\ 1-\frac{1}{4} e^{-2x} & x\ge 0 \end{cases}$$

a:



b: $P(X\le 2) = F_X(2) = 1-\frac{1}{4}e^{-4}$

$P(X=0) = \frac{3}{4}$

$P(X<0) = 0$

$P(2<X<6) = F_X(6) - F_X(2) = \left(1-\frac{1}{4}e^{-12}\right) - \left(1-\frac{1}{4}e^{-4}\right)$

$= \frac{1}{4}\left(e^{-4} - e^{-12}\right)$

$P(X>10) = 1 - P(X\le 10) = 1 - F_X(10) = 1 - \left(1-\frac{1}{4}e^{-20}\right) = \frac{1}{4}e^{-20}$

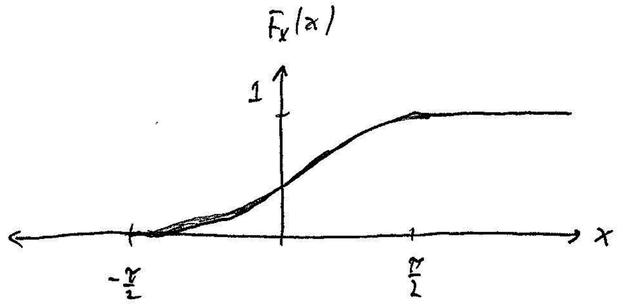
Question 74:

$$F_X(x) = \begin{cases} 0 & x \le -\frac{\pi}{2} \\ c(1+\sin(x)) & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & x \ge \frac{\pi}{2} \end{cases}$$

a: If c is 1 $F_X(\frac{\pi}{2} - \varepsilon) > 1$

This is not possible, since it would imply the probability of the random variable taking on a range of the sample space is > 1.

b.



$$\because F_X\left(-\frac{\pi}{2}\right) = F_X\left(-\frac{\pi^-}{2}\right) = F_X\left(-\frac{\pi^+}{2}\right) = 0$$

$$F_X\left(\frac{\pi}{2}\right) = F_X\left(\frac{\pi^-}{2}\right) = F_X\left(\frac{\pi^+}{2}\right) = 1$$

$\therefore X$ is a continuous random variable

c: for $x < -\frac{\pi}{2}$, $f_X(x) = 0$

for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $f_X(x) = \frac{d}{dx}\frac{1}{4}(1+\sin(x)) = \frac{1}{4}\cos(x)$

$P(X = \frac{\pi}{2}) = 1 - \frac{1}{4}\left(1 + \sin\left(\frac{\pi}{2}\right)\right) = 1 - \frac{1}{2} = \frac{1}{2}$

$f_X(x) = \frac{1}{4}\cos(x)\left(\mathcal{U}(x+\frac{\pi}{2}) - \mathcal{U}(x-\frac{\pi}{2})\right) + \frac{1}{2}\delta(x-\frac{\pi}{2})$

Question 75:

$$f_X(x) = \frac{1}{2} e^{-|x|}$$

$$Y = \min(X, 0)$$

for $y < 0$, $P(Y \leq y) = F_Y(y) = P(X \leq y) = F_X(y)$

$$= \int_{-\infty}^{y} \frac{1}{2} e^{x} \, dx = \frac{1}{2} e^{x} \Big|_{-\infty}^{y} = \frac{1}{2} e^{y}$$

$$P(Y=0) = P(X \geq 0) = \int_0^{\infty} \frac{1}{2} e^{-x} \, dx = -\frac{1}{2} e^{-x} \Big|_0^{\infty} = \frac{1}{2}$$

$$F_Y(y) = \begin{cases} \frac{1}{2} e^{y} & y < 0 \\ 1 & y \geq 0 \end{cases}$$

$$f_Y(y) = \frac{1}{2} e^{y} \, u(-y) + \frac{1}{2} \delta(y)$$

Question 76:

$$X \sim U(0,1)$$

$$Y = -\ln(X)$$

$$P(Y \leq y) = F_Y(y) = P(-\ln(X) \leq y) = P\left(\frac{1}{X} \leq e^{y}\right) = P(X \geq e^{-y})$$

$$= 1 - \int_0^{e^{-y}} (1) \, dx = 1 - x \Big|_0^{e^{-y}} = 1 - e^{-y}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y} & y \geq 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ e^{-y} & y \geq 0 \end{cases} \qquad Y \sim \text{Exponential}(1)$$

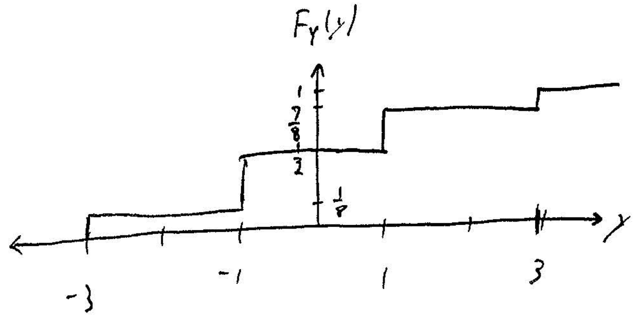
Question 77:

$P(Y=3) = P(Y=-3) = \frac{1}{8}$

~~$P(Y=2) = P(Y=-2) = \frac{3}{8}$~~

$P(Y=1) = P(Y=-1) = \frac{3}{8}$

a:

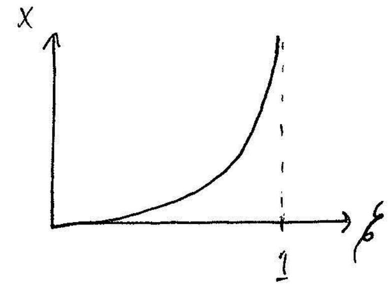


b: $P(|Y| < y) = P(-y < Y < y) = F_Y(y) - F_Y(-y)$

Question 78:

$\xi \sim U(0,1) \qquad X = (1-\xi)^{-\frac{1}{2}}$

a:



$$F_\xi(x) = \begin{cases} 0 & x<0 \\ x & 0\le x\le 1 \\ 1 & x>1 \end{cases}$$

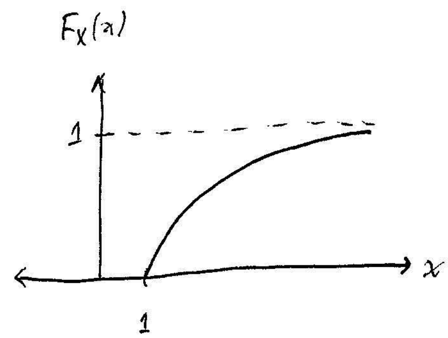
b: $F_X(x) = P(X\le x) = P\left((1-\xi)^{-\frac{1}{2}} \le x\right) = P\left(\frac{1}{1-\xi} \le x^2\right)$

$= P\left(1-\xi \ge \frac{1}{x^2}\right) = P\left(\xi \le 1-\frac{1}{x^2}\right)$

$= F_\xi\left(1-\frac{1}{x^2}\right)$

$$= \begin{cases} 0 & 1-\frac{1}{x^2}<0 \\ 1-\frac{1}{x^2} & 0\le 1-\frac{1}{x^2}\le 1 \\ 1 & 1-\frac{1}{x^2} \ge 1 \end{cases}$$

$$= \begin{cases} 0 & x<1 \\ 1-\frac{1}{x^2} & 1\le x \end{cases}$$



c: $P(X>1) = 1 - F_X(1) = 1$

$P(5<X<7) = F_X(7) - F_X(5) = \left(1-\frac{1}{49}\right) - \left(1-\frac{1}{25}\right) = \frac{1}{25} - \frac{1}{49}$

$P(X\le 20) = F_X(20) = 1 - \frac{1}{400} = \frac{399}{400}$

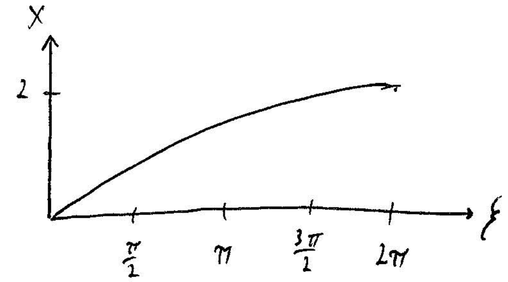
Question 79:

$X = 2\sin(\xi/4)$

$\xi \sim \mathrm{Uniform}(0, 2\pi)$

$$F_\xi(x) = \begin{cases} 0 & x<0 \\ \frac{x}{2\pi} & 0 \le x \le 2\pi \\ 1 & x > 2\pi \end{cases}$$

a:

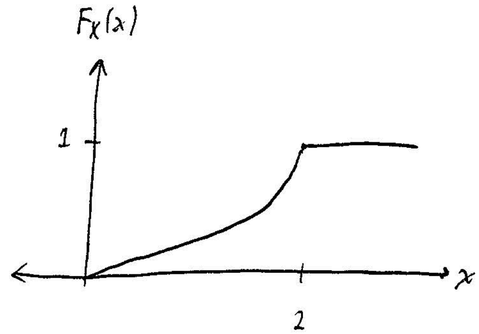


b: $P(X \le x) = F_X(x) = P(2\sin(\xi/4) \le x) = P\left(\frac{\xi}{4} \le \mathrm{asin}\left(\frac{x}{2}\right)\right) = P\left(\xi \le 4\,\mathrm{asin}\left(\frac{x}{2}\right)\right)$

$= F_\xi\left(4\,\mathrm{asin}\left(\frac{x}{2}\right)\right)$

$$= \begin{cases} 0 & 4\,\mathrm{asin}\left(\frac{x}{2}\right) < 0 \\ \frac{1}{2\pi} 4\,\mathrm{asin}\left(\frac{x}{2}\right) & 0 < 4\,\mathrm{asin}\left(\frac{x}{2}\right) < 1 \\ 1 & 4\,\mathrm{asin}\left(\frac{x}{2}\right) > 1 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \frac{2}{\pi}\,\mathrm{asin}\left(\frac{x}{2}\right) & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$



c: $P(X > 1) = 1 - P(X \le 1) = 1 - \frac{2}{\pi}\,\mathrm{asin}\left(\frac{1}{2}\right) = \frac{2}{3}$

$P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = F_X\left(\frac{1}{2}\right) - F_X\left(-\frac{1}{2}\right) = F_X\left(\frac{1}{2}\right) = \frac{2}{\pi}\,\mathrm{asin}\left(\frac{1}{4}\right) = 0.1609$

$P\left(X \le \frac{1}{\sqrt{2}}\right) = F_X\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\pi}\,\mathrm{asin}\left(\frac{1}{2\sqrt{2}}\right) = 0.2301$

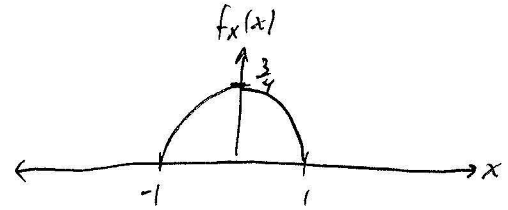
Question 80:

$$f_X(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0 & \text{ow} \end{cases}$$

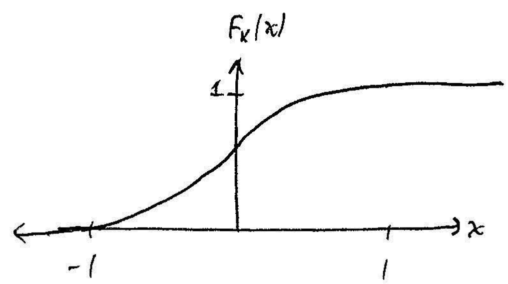
a: for $-1 \le x \le 1$, $F_X(x) = \int_{-\infty}^{x} c(1-s^2)\,ds = c\left(s - \frac{1}{3}s^3\right)\Big|_{-1}^{x}$

$$= c\left[\left(x - \frac{1}{3}x^3\right) - \left(-1 + \frac{1}{3}\right)\right] = c\left(x - \frac{1}{3}x^3 + \frac{2}{3}\right)$$

$$F_X(1) = 1 \Rightarrow c = \frac{1}{1 - \frac{1}{3} + \frac{2}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$



b:



c: $P(X=0) = 0$

$$P\left(0 < X < \frac{1}{2}\right) = F_X\left(\frac{1}{2}\right) - F_X(0) = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right) + \frac{2}{3}\right) - \frac{1}{2}$$

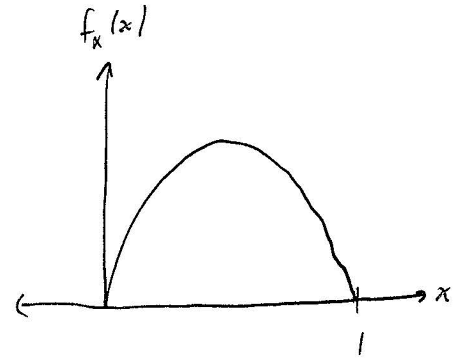
$$= \frac{3}{4}\left(\frac{12 - 1 + 16}{24}\right) - \frac{1}{2} = \frac{27}{32} - \frac{1}{2} = \frac{27-16}{32} = \frac{11}{32}$$

$$P\left(\left|X - \frac{1}{2}\right| < \frac{1}{4}\right) = P\left(\frac{1}{4} < X < \frac{3}{4}\right) = F_X\left(\frac{3}{4}\right) - F_X\left(\frac{1}{4}\right)$$

$$= \frac{3}{4}\left(\frac{3}{4} - \frac{1}{3}\left(\frac{27}{64}\right) + \frac{2}{3}\right) - \frac{3}{4}\left(\frac{1}{4} - \frac{1}{3}\left(\frac{1}{64}\right) + \frac{2}{3}\right) \approx 0.2734$$

Question 81:

$$f_X(x) = \begin{cases} cx(1-x^2) & 0 \le x \le 1 \\ 0 & ow \end{cases}$$



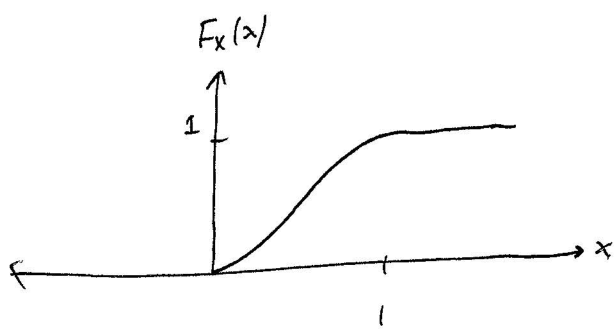
a: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow c\int_0^1 (x - x^3)\,dx = 1$

$$c\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1 = c\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = c\tfrac{1}{4} \Rightarrow c = 4$$

$$F_X(x) = \begin{cases} 0 & x<0 \\ 4\int_0^x (s - s^3)\,ds & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$= \begin{cases} 0 & x<0 \\ 2x^2 - x^4 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

b:



c: $P(0 < X < \tfrac{1}{2}) = F_X(\tfrac{1}{2}) - F_X(0) = 2(\tfrac{1}{4}) - \tfrac{1}{16} = \tfrac{7}{16}$

$P(X=1) = 0$

$P(\tfrac{1}{4} < X < \tfrac{1}{2}) = F_X(\tfrac{1}{2}) - F_X(\tfrac{1}{4}) = \tfrac{7}{16} - \left[(2)\left(\tfrac{1}{16}\right) - \tfrac{1}{256}\right]$

$$= \frac{7}{16} - \left[\frac{32-1}{256}\right] = \frac{112-31}{256} = \frac{81}{256}$$

Question 82:

$$Y = 2X + 3$$

$$F_Y(y) = P(Y \le y) = P(2X+3 \le y) = P\left(X \le \tfrac{1}{2}(y-3)\right) = F_X\left(\tfrac{1}{2}(y-3)\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2} f_X\left(\frac{1}{2}y - \frac{3}{2}\right)$$

Question 83:

$$X \sim N(2,4)$$

$$Y = \max(X, 0)$$

$$P(X<0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi 4}} e^{-\frac{(x-2)^2}{8}} \, dx$$

$$\cancel{f_Y(y) = \begin{cases} \frac{1}{\sqrt{8\pi}} e^{-\frac{(y-2)^2}{8}} & y>0 \\ P(X<0) & y=0 \end{cases}}$$

$$f_Y(y) = \frac{1}{\sqrt{8\pi}} e^{-\frac{(y-2)^2}{8}} u(y) + P(X<0)\delta(y)$$

Question 84:

$U \sim \text{Uniform}(0,1)$

$X = U^n$

$$F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \le u \le 1 \\ 1 & u > 1 \end{cases}$$

$$F_X(x) = P(X \le x) = P(U^n \le x) = P(U \le \sqrt[n]{x})$$

$$= F_u(\sqrt[n]{x}) = \begin{cases} 0 & x < 0 \\ \sqrt[n]{x} & 0 \le x \le 1 \\ 1 & \text{ow} \end{cases}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{n} x^{\frac{1}{n}-1} & 0 \le x \le 1 \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} \frac{1}{n} x^{\frac{1-n}{n}} & 0 \le x \le 1 \\ 0 & \text{ow} \end{cases}$$