ECE 302-003 Homework #6 Solution

Fall 2023

Question 62:

$$F(x) = \int_0^x f(s)\,ds \qquad g(x) = F(h(x))$$

$\frac{d}{dx} F(x) = f(x)$ by the fundamental theorem of calculus

$$\frac{d}{dx} g(x) = F'(h(x))\,h'(x)$$
$$= f(h(x)) \frac{d}{dx} h(x)$$

Question 63 :

$S_X = [2, \pi]$ #

$$f_X(x) = \begin{cases} \frac{1}{\pi - 2} & \text{if } 2 \le x \le \pi \\ 0 & \text{otherwise.} \end{cases}$$ #

$$E(X) = \int_2^\pi x \frac{1}{\pi-2}\,dx = \frac{1}{\pi-2} \frac{1}{2} x^2 \Big|_2^\pi = \frac{1}{\pi-2} \frac{1}{2}(\pi^2 - 2^2) = \frac{\pi+2}{2}$$ #

$$E(X^2) = \int_2^\pi x^2 \frac{1}{\pi-2}\,dx = \frac{1}{\pi-2} \frac{1}{3} x^3 \Big|_2^\pi = \frac{1}{\pi-2} \frac{1}{3}(\pi^3 - 2^3) = \frac{\pi^2+2\pi+4}{3}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{\pi^2+2\pi+4}{3} - \frac{\pi^2+4\pi+4}{4} = \frac{\pi^2 - 4\pi + 4}{12}$$
$$= \frac{(\pi-2)^2}{12}$$ #

P(the integer part of X is an even number)

$$= P(2 \le X < 3) = \int_2^3 \frac{1}{\pi-2}\,dx = \frac{1}{\pi-2}$$ #

Question 64:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & ow \end{cases} \qquad S = [0, \infty)$$

$$= \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & ow \end{cases}$$

$$E[X] = \int_0^\infty x\, 3e^{-3x}\, dx \qquad \begin{array}{ll} u = x & dv = 3e^{-3x} dx \\ du = dx & v = -e^{-3x} \end{array}$$

$$= -xe^{-3x}\Big|_0^\infty + \int_0^\infty e^{-3x}\, dx$$

$$= 0 - 0 - \frac{1}{3}e^{-3x}\Big|_0^\infty = -\frac{1}{3}(0-1) = \frac{1}{3}$$

$$E[X^2] = \int_0^\infty x^2\, 3e^{-3x}\, dx \qquad \begin{array}{ll} u = x^2 & dv = 3e^{-3x} dx \\ du = 2x\, dx & v = -e^{-3x} \end{array}$$

$$= -x^2 e^{-3x}\Big|_0^\infty + 2\int_0^\infty x e^{-3x}\, dx$$

$$= 0 - 0 + \frac{2}{3} E[X] = \frac{2}{3}\left(\frac{1}{3}\right) = \frac{2}{9}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$Y = \lfloor X \rfloor = P(0 \leq X < 1) + P(2 \leq X < 3) + \dots = \sum_{k=0}^\infty \left[\int_{2k}^{2k+1} 3e^{-3x}\, dx\right] = \sum_{k=0}^\infty -\left(e^{-3(2k+1)} - e^{-6k}\right)$$

$$= \sum_{k=0}^\infty e^{-6k}(1-e^{-3}) = \frac{1-e^{-3}}{1-e^{-6}} = \frac{1}{1+e^{-3}}$$

Question 65

$S = (-\infty, \infty)$ #

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-2)^2}{18}}$$ #

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-2)^2}{18}} dx$$

$$= \int_{-\infty}^{\infty} (y+2) \frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}} dy$$

Let $y = x-2$

$dy = dx$

$$= \int_{-\infty}^{\infty} y \frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}} dy + 2\underbrace{\int_{-\infty}^{\infty} \frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}} dy}_{=1}$$

$$= \left[\frac{-3}{\sqrt{2\pi}} e^{-\frac{y^2}{18}} \right]_{-\infty}^{\infty} + 2$$

$$= 0 - 0 + 2 = 2$$ #

Question 66:

$$f_X(x) = \frac{3}{2} e^{-3|x|}$$

$$P(Y=k) = P(k \le |X| < k+1) = P(k \le X < k+1) + P(-(k+1) < X \le -k)$$

$$= \int_k^{k+1} \frac{3}{2} e^{-3|x|} dx + \int_{-(k+1)}^{-k} \frac{3}{2} e^{-3|x|} dx$$

$$= 3\int_k^{k+1} e^{-3x} dx = -e^{-3x}\Big|_k^{k+1}$$

$$= -\left(e^{-3(k+1)} - e^{-3k}\right) = e^{-3k}\left(1 - e^{-3}\right) \qquad \text{for } k = 0, 1, 2, \ldots$$

Question 67:

$P(Y_i = 1) = p \qquad Y_i \text{ iid} \qquad X = Y_1 + Y_2 + \cdots + Y_n$

a: The sample space contains all possible (2^n) combinations of 1 & 0 that make an n-bit long string.

b: $P(Y_1, \ldots, Y_n) = p^k (1-p)^{n-k}$ for k 1's

c: $P(Y_k = 1 \mid X = 1) = \dfrac{P(Y_k = 1 \cap X = 1)}{P(X=1)}$

Note: There is exactly one outcome for $Y_k = 1$ and $X = 1$

$P(Y_k = 1 \cap X = 1) = p(1-p)^{n-1}$

$P(X=1) = n\,p(1-p)^{n-1}$

$\hookrightarrow = \dfrac{p(1-p)^{n-1}}{np(1-p)^{n-1}} = \dfrac{1}{n}$

d: $P(Y_k = 1 \mid X = 2) = \dfrac{P(Y_k = 1 \cap X = 2)}{P(X=2)}$

$P(Y_k = 1 \cap X = 2) = (n-1)\,p^2(1-p)^{n-2}$

$P(X=2) = \binom{n}{2} p^2 (1-p)^{n-2}$

$\hookrightarrow = \dfrac{(n-1)p^2(1-p)^{n-2}}{\binom{n}{2} p^2(1-p)^{n-2}} = \dfrac{n-1}{\binom{n}{2}} = \dfrac{n-1}{\frac{n!}{(n-2)!\,2!}} = \dfrac{2\,(n-2)!\,(n-1)}{n\,(n-1)\,(n-2)!} = \dfrac{2}{n}$

Question 68:

Let X = # of photons detected

Y = on / off

$$P(X=k \mid Y=\text{on}) = \frac{e^{-\alpha_1}}{k!}\alpha_1^k \qquad \alpha_1 = 200$$

$$P(X=k \mid Y=\text{off}) = \frac{e^{-\alpha_2}}{k!}\alpha_2^k \qquad \alpha_2 = 50$$

a: $P(Y=\text{on}) = P(Y=\text{off}) = \frac{1}{2}$

$$P(Y=\text{on} \mid X=125) = \frac{P(Y=\text{on} \cap X=125)}{P(X=125)} = \frac{\frac{1}{2}\frac{e^{-\alpha_1}}{(125!)}(200)^{125}}{\frac{1}{2}\left[\frac{e^{-\alpha_2}}{125!}50^{125} + \frac{e^{-\alpha_1}}{125!}200^{125}\right]}$$

$$= \frac{e^{-200}(200)^{125}}{e^{-50}(50)^{125} + e^{-200}(200)^{125}} = 1 - 7.7032\times10^{-11}$$

b: $P(Y=\text{on} \mid X=125) \gg \frac{1}{2} \Rightarrow$ conclude it is ~~off~~ on

c: $$\frac{e^{-\alpha_1}\alpha_1^k}{e^{-\alpha_1}\alpha_1^k + e^{-\alpha_2}\alpha_2^k} \geq \frac{1}{2} \Rightarrow e^{-\alpha_1}\alpha_1^k \geq \frac{1}{2}e^{-\alpha_1}\alpha_1^k + \frac{1}{2}e^{-\alpha_2}\alpha_2^k$$

$$e^{-\alpha_1}\alpha_1^k \geq e^{-\alpha_2}\alpha_2^k$$

$$-\alpha_1 + k\ln\alpha_1 \geq -\alpha_2 + k\ln\alpha_2$$

$$k(\ln\alpha_1 - \ln\alpha_2) \geq \alpha_1 - \alpha_2$$

$$k \geq \frac{\alpha_1 - \alpha_2}{\ln\frac{\alpha_1}{\alpha_2}} = \frac{200-50}{\ln 4} = 108.2$$

any count above 108 $\Rightarrow$ the laser was on

Question 69:

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{ow} \end{cases}$$

$$P(X^2 < 3 \mid X > 1) = P(-\sqrt{3} < X < \sqrt{3} \mid X > 1)$$

$$= \frac{P((-\sqrt{3} < X < \sqrt{3}) \cap X > 1)}{P(X > 1)} = \frac{P(1 < X < \sqrt{3})}{P(X > 1)}$$

$$P(1 < X < \sqrt{3}) = \int_1^{\sqrt{3}} 2e^{-2x}\,dx = -e^{-2x}\Big|_1^{\sqrt{3}} = e^{-2} - e^{-2\sqrt{3}}$$

$$P(X > 1) = \int_1^{\infty} 2e^{-2x}\,dx = -e^{-2x}\Big|_1^{\infty} = e^{-2}$$

$$= \frac{e^{-2} - e^{-2\sqrt{3}}}{e^{-2}} = 1 - e^{-2(\sqrt{3}-1)}$$

Question 70:

a: $X \sim Exp(\frac{5}{6})$ in minutes

$$P(X>3) = \int_3^\infty \frac{5}{6} e^{-\frac{5}{6}x} dx$$

$$= -e^{-\frac{5}{6}x}\Big|_3^\infty = e^{-\frac{5}{2}}$$

b: $$P(Y=k) = \frac{e^{-\frac{5}{6}(3)}}{k!}\left(\frac{5}{6}(3)\right)^k$$

$$P(Y=0) = \frac{e^{-\frac{5}{2}}}{0!}\left(\frac{5}{2}\right)^0 = e^{-\frac{5}{2}}$$

c: $P(X>3) = P(Y=0)$ since they are the same event (no meteors in 3 minutes)