ECE 302-003, Homework #6
It is a self-exercise. No need to turn in the homework.

`https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html`

Review of Calculus: The chain rule.

Question 62: Consider the following functions.

$$F(x) = \int_0^x f(s)ds$$
$$g(x) = F(h(x)) = \int_0^{h(x)} f(s)ds$$

Express $\frac{d}{dx}F(x)$ and $\frac{d}{dx}g(x)$ using $f(x)$ and $h(x)$.

Question 63: [Basic] Write down the sample space and weight assignment (pdf) for a uniform random variable X with parameters $a = 2$ and $b = \pi$. Compute its mean $E(X)$, variance $\text{Var}(X)$ and the probability $P(\text{the integer part of } X \text{ is an even number})$.

Question 64: [Intermediate/Exam Level] Write down the sample space and weight assignment (pdf) for an exponential random variable X with parameters $\lambda = 3$. Compute its mean $E(X)$, variance $\text{Var}(X)$ and the probability $P(\text{the integer part of } X \text{ is an even number})$.

Question 65: [Basic] Write down the sample space and weight assignment (pdf) for a Gaussian random variable X with parameters $\mu = 2$ and $\sigma = 3$. Compute its mean $E(X)$.

Question 66: [Intermediate/Exam Level] (Q42). Consider a continuous random variable X with the following pdf $f_X(x)$:

$$f_X(x) = 1.5e^{-3|x|} \quad \text{for all } x \tag{1}$$

Consider a discrete "quantizer" Y of the magnitude of X as follows. For any X, if $k \leq |X| < k + 1$, then $Y = k$. For example, if the X value is $-\pi$, then $Y = 3$ since $3 \leq |-\pi| < 4$. If the X value is 1.25, then $Y = 1$. Find the pmf of the discrete variable Y. Namely, find $P(Y = k)$ for $k = 0, 1, 2 \cdots$. What type of random variables is Y?

Question 67: [Intermediate/Exam Level]

Let $X = Y_1 + Y_2 + \cdots + Y_n$ be a binomial random variable that results from the summation of n independent Bernoulli random variables Y_1 to Y_n. The success probability of Y_i is p for $i = 1, \cdots, n$.

1. What is the sample space of $(Y_1, \ldots, Y_n)$? (Hint: The sample space is too large to enumerate the outcomes exhaustively. You should describe the sample space by something like "The sample space contains all combinations of ...".)

2. What is the weight assignment for each outcome of $(Y_1, \ldots, Y_n)$?

3. What is the conditional probability $P(Y_k = 1|X = 1)$? (Hint: k is simply a fixed index. You can first try to answer the question $P(Y_3 = 1|X = 1)$.)

4. What is the conditional probability $P(Y_k = 1|X = 2)$?

Question 68: [Intermediate/Exam Level] If a laser is "ON", from the empirical data, it will emit 200.0 photons per millisecond. If it is "OFF", from the empirical data, it will still emit 50.0 photons per millisecond due to the thermal noise. (Using the Poisson distribution, the above two statements are specifying the conditional probabilities $P(\#\text{ of photons}|\text{state (ON/OFF)})$.)

1. Suppose the laser is equally likely to be "ON" or "OFF", and we have observed 125 photons in a millisecond period. What is the conditional probability $P(\text{on}|\#\text{ of photons} = 125)$? (Hint: You need to use the Poisson distribution. A calculator is also necessary to compute this value.)

2. If you are the observer, do you think the laser is "ON" or "OFF"? (Note that 125 is exactly the average of 200 and 50.)

3. [Optional] Suppose you observe $k = 0$ photons in 1 millisecond. Is the laser more likely to be "ON" or "OFF"? Repeat the same question for $k = 1, 2, \cdots, 200$. Suppose you want to *set the threshold* K for the photon count X such that if $X \geq K$, the photon detector outputs "1" and when $X < K$, the photon detector outputs "0." How would you set the threshold value K? You should ask yourself why this problem is important, especially from the optical communication perspective.

Question 69: [Basic] Suppose X is exponentially distributed with parameter $\lambda = 2$. Find out the conditional probability $P(X^2 < 3|X > 1)$.

Question 70: [Basic]

1. Suppose the arrival time X of the next meteor is exponential distributed, and the average number of arrivals in one hour is 50. What is the probability that you have waited for 3 minutes without seeing a shooting star, namely, what is $P(X > 3)$.

2. Let Y denote the number of arrivals within the first 3 minutes, which is Poisson distributed. Find out the probability that $P(Y = 0)$.

3. [Optional] Is $P(Y = 0) = P(X > 3)$? Why?