ECE 302-003 Homework #5 Solution

Fall 2023

Question 49:

$$P_k = \frac{c}{k^2} \qquad k = 1, 2, \cdots$$

a: $\displaystyle\sum_{k=1}^{2} \frac{c}{k^2} = c\left(1+\frac{1}{4}\right) = \frac{5}{4}c$

Note: $\displaystyle\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\displaystyle\sum_{k=1}^{3} \frac{c}{k^2} = c\left(1+\frac{1}{4}+\frac{1}{9}\right) = \left(\frac{36+9+4}{36}\right)c = \frac{49}{36}c$

$\displaystyle\sum_{k=1}^{4} \frac{c}{k^2} = c\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{64}\right) = 1.3767c$

$\vdots$

$$c = \frac{6}{\pi^2}$$

b: $P(X \geq 4) = 1 - P(X \leq 3)$

$\displaystyle = 1 - \frac{6}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}\right) = 1 - \frac{6}{\pi^2}\left(\frac{49}{36}\right) = 1 - \frac{49}{6\pi^2}$

c: $\displaystyle P(6 \leq X \leq 8) = p\,\frac{6}{\pi^2}\left(\frac{1}{36}+\frac{1}{49}+\frac{1}{64}\right)$

Question 50:

a: $P(Y=y) = \begin{cases} \frac{4}{10} & y=2 \\ \frac{3}{10} & y=1 \\ \frac{2}{10} & y=0 \\ \frac{1}{10} & y=-1 \\ 0 & \text{ow} \end{cases}$

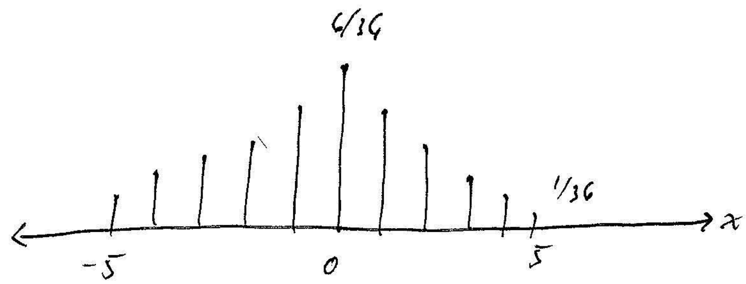
b: $P(Y=2) = \frac{4}{10}$

c: $P(Y>0) = P(Y=1) + P(Y=2) = \frac{7}{10}$

Question 51:

a:

$P_X(x) = \begin{cases} \frac{1}{36} & x = \pm 5 \\ \frac{2}{36} & x = \pm 4 \\ \frac{3}{36} & x = \pm 3 \\ \frac{4}{36} & x = \pm 2 \\ \frac{5}{36} & x = \pm 1 \\ \frac{6}{36} & x = 0 \\ 0 & \text{ow} \end{cases}$



b:

$P(|X| \leq k) = \begin{cases} \frac{6}{36} & k<1 \\ \frac{16}{36} & k<2 \\ \frac{24}{36} & k<3 \\ \frac{30}{36} & k<4 \\ \frac{34}{36} & k<5 \\ 1 & k \geq 5 \end{cases}$

Question 52:

$$E[Y] = \frac{4}{10}(2) + \frac{3}{10}(1) + \frac{2}{10}(0) + \frac{1}{10}(-1)$$

$$= \frac{8+3-1}{10} = 1$$

$$E[Y^2] = \frac{4}{10}(4) + \frac{3}{10}(1) + \frac{2}{10}(1) + \frac{1}{10}(1)$$

$$= \frac{16+3+1}{10} = 2$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = 2 - (1)^2 = 1$$

Question 53:

$P_X(k) = \binom{4}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{4-k} \quad k = 0,1,2,3,4 \qquad S = \{0,1,2,3,4\}$

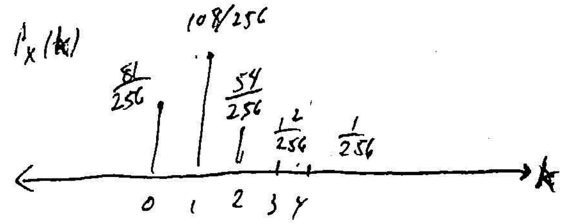
$P_X(0) = (1)(1)\left(\frac{3}{4}\right)^4 = \frac{81}{256}$

$P_X(1) = (4)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3 = \frac{27}{64} = \frac{108}{256}$

$P_X(2) = (6)\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{6(9)}{4(256)} = \frac{54}{256}$

$P_X(3) = (4)\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) = \frac{12}{256}$

$P_X(4) = (1)\left(\frac{1}{4}\right)^4(1) = \frac{1}{256}$



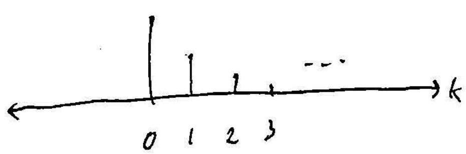
$E[X] = (0)\frac{81}{256} + (1)\frac{108}{256} + (2)\frac{54}{256} + (3)\frac{12}{256} + (4)\frac{1}{256} = \frac{108+108+36+4}{256} = \frac{256}{256} = 1$

$E[X^2] = (0)\frac{81}{256} + (1)\frac{108}{256} + (4)\frac{54}{256} + (9)\frac{12}{256} + (16)\frac{1}{256} = \frac{108+216+108+16}{256} = \frac{448}{256} = \frac{7}{4}$

$Var(X) = E[X^2] - (E[X])^2 = \frac{7}{4} - 1 = \frac{3}{4}$

Question 54:

$S = \{1,2,3,\ldots\} \qquad P_X(k) = \frac{1}{3}\left(\frac{2}{3}\right)^k$



$E[X] = \sum_{k=1}^{\infty} \frac{k}{3}\left(\frac{2}{3}\right)^k = \frac{1}{3}\frac{\frac{2}{3}}{\left(1-\frac{2}{3}\right)^2} = \frac{1}{3}\frac{\frac{2}{3}}{\frac{1}{9}} = 2$

$E[X^2] = \sum_{k=1}^{\infty} k^2 \frac{1}{3}\left(\frac{2}{3}\right)^k = \frac{1}{3}\frac{2}{3}\frac{1+\frac{2}{3}}{\left(1-\frac{2}{3}\right)^3} = \frac{1}{3}\frac{2}{3}\frac{\frac{5}{3}}{\frac{1}{27}} = 10$

$Var(X) = E[X^2] - (E[X])^2 = 10 - 4 = 6$

$P(X \le 5 \text{ \& is prime}) = P(X=2) + P(X=3) + P(X=5) = \frac{1}{3}\left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5\right) = \frac{1}{3}\left(\frac{2}{3}\right)^2\left(1 + \frac{2}{3} + \frac{8}{27}\right)$

$= \frac{4}{27}\left(\frac{27+18+8}{27}\right) = \frac{4(53)}{729} = \frac{212}{729}$

Question 55:

$$S = \{0, 1, 2, \ldots\} \qquad p_X(k) = \frac{2^k}{k!} e^{-2\cancel{k}}$$

$$E[X] = \sum_{k=0}^{\infty} k \frac{2^k}{k!} e^{-2\cancel{k}} = \sum_{k=0}^{\infty} \frac{2^k}{(k-1)!} e^{-2\cancel{k}} \qquad \text{Let } l = k-1,\ k = l+1$$

$$= \sum_{l=0}^{\infty} \frac{2^{l+1}}{l!} e^{-2\cancel{(l+1)}} = 2\cancel{\ } \sum_{l=0}^{\infty} \frac{2^l}{l!} e^{-2l} = 2\cancel{\ }$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{2^k}{k!} e^{-2} = \sum_{k=1}^{\infty} k \frac{2^k}{(k-1)!} e^{-2} \qquad \text{Let } l = k-1, k = l+1$$

$$= \sum_{l=0}^{\infty} (l+1) \frac{2^{(l+1)}}{l!} e^{-2} = 2\left[\sum_{l=0}^{\infty} l \frac{2^l}{l!} e^{-2} + \sum_{l=0}^{\infty} \frac{2^l}{l!} e^{-2}\right] = 2\left[E[X] + 1\right]$$

$$= 2(2+1) = 6$$

$$Var(X) = E[X^2] - (E[X])^2 = 6 - 4 = 2$$

$$P(X \leq 5 \text{ \& is prime}) = P(X=2) + P(X=3) + P(X=5)$$

$$= \frac{4}{2} e^{-2} + \frac{8}{6} e^{-2} + \frac{32}{120} e^{-2} = \frac{240 + 160 + 32}{120} e^{-2} = \frac{432}{120} e^{-2} = \frac{18}{5} e^{-2}$$

Question 56:

$$S_Y = \{0, 1, 2\}$$

$$p_Y(y) = \begin{cases} \frac{1}{4} & y = 0, 2 \\ \frac{1}{2} & y = 1 \\ 0 & ow \end{cases}$$

Y is binomial, $n = 2$, $p = \frac{1}{2}$

Question 57:

$$S = \{(0,0,0,0),\ (0,0,0,1),\ (0,0,1,0),\ (0,0,1,1),\ (0,1,0,0),\ (0,1,0,1),\ (0,1,1,0),$$
$$(0,1,1,1),\ (1,0,0,0),\ (1,0,0,1),\ (1,0,1,0),\ (1,0,1,1),\ (1,1,0,0),\ (1,1,0,1),\ (1,1,1,0),$$
$$(1,1,1,1)\}$$

Each outcome is equally likely $\left(\frac{1}{16}\right)$

$X = \max\left(f(z_1, z_2),\ f(w_1, w_2)\right)$

$S_X = \{0, 1, 2\}$

$$P_X(x) = \begin{cases} \frac{1}{16} & x = 0 \\ \frac{8}{16} & x = 1 \\ \frac{7}{16} & x = 2 \\ 0 & ow \end{cases}$$

Question 58:

a: $P_N(n) = \begin{cases} (0.01)(0.99)^n & n = 0, 1, 2, \ldots \\ 0 & ow \end{cases}$

b: $E[N] = \sum_{n=0}^{\infty} n\,(0.01)(0.99)^n = \cancel{\sum_{n=0}^{\infty} n(0.01)(0.99)^{n-1}}$

$$= 0.01 \sum_{n=0}^{\infty} n(0.99)^n = 0.01\,\frac{0.99}{(1-0.99)^2} = \frac{0.99}{(0.01)\cancel{(0.01)}} = 99$$

c: $P(N \geq 1000) = \sum_{n=1000}^{\infty} p(1-p)^n$ Let $k = n - 1000$, $n = k + 1000$

$$= p\sum_{k=0}^{\infty}(1-p)^{k+1000} = p(1-p)^{1000}\sum_{k=0}^{\infty}(1-p)^k = \frac{p(1-p)^{1000}}{1-(1-p)} = (1-p)^{1000}$$

For $p = 0.01$: $p(N \geq 1000) = 4.3171 \times 10^{-5}$

For $P(N \geq 1000) > 0.99$: $(1-p)^{1000} > 0.99$

$$1000 \ln(1-p) > \ln(0.99)$$

$$\ln(1-p) > \frac{\ln(0.99)}{1000} \qquad 1 - p > e^{\left(\frac{\ln(0.99)}{1000}\right)}$$

$$p < 1 - e^{\left(\frac{\ln(0.99)}{1000}\right)}$$

$$p < 1.005 \times 10^{-5}$$

Question 59:

$$p_M(k) = \begin{cases} p(1-p)^k & k = 0, 1, 2, \ldots \\ 0 & ow \end{cases}$$

$$P(M \geq k) = \sum_{n=k}^{\infty} p(1-p)^n \qquad \text{Let } l = n-k, \quad n = l+k$$

$$= p \sum_{l=0}^{\infty} (1-p)^{l+k} = p(1-p)^k \frac{1}{1-(1-p)} = (1-p)^k$$

$$P(M \geq k+j \mid M \geq j) = \frac{P(M \geq k+j \ \& \ M \geq j)}{P(M \geq j)} = \frac{P(M \geq k+j)}{P(M \geq j)} = \frac{(1-p)^{k+j}}{(1-p)^j} = (1-p)^k$$

$$= P(M \geq k)$$

Question 60:

$$P_N(k) = \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{for } k = 0, 1, 2, \ldots \qquad \alpha = \frac{6000}{min}$$

a: $\frac{6000}{min}\left(\frac{min}{60\ s}\right) = \frac{100}{s}$

$$\alpha = \frac{100}{s}\left(\frac{s}{1000\ ms}\right)(100\ ms) = 10 \text{ requests}$$

$$P_N(0) = \frac{(10)^0}{0!} e^{-(10)} = e^{-10}$$

b: $P(5 \le N \le 10) = \sum_{n=5}^{10} \frac{10^k}{k!} e^{-10} = e^{-10}\left(\frac{10^5}{5!} + \frac{10^6}{6!} + \frac{10^7}{7!} + \frac{10^8}{8!} + \frac{10^9}{9!} + \frac{10^{10}}{10!}\right) = 0.5538$

c: $P(N > 15) = 1 - P(N \le 15) = 1 - \sum_{n=0}^{15} \frac{(10)^n}{n!} e^{-10} = 0.0487$

Question 61:

Let X be the number of dead pixels, $p = 10^{-5}$, $q = 1-p$, $N = (1000)(750)$

$$P_X(k) = \begin{cases} \binom{N}{k} p^k q^{N-k} & k = 0, 1, \ldots, N \\ 0 & \text{ow} \end{cases}$$

$$P(X \le 15) = \sum_{k=0}^{15} \binom{N}{k} p^k q^{N-k} = 0.9954$$