ECE 302-003, Homework #5
Due date: Wednesday 10/04/2023, 11:59pm;
Submission via Gradescope

https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html

Question 49: [Basic] Problem 3.13. Hint: You should first try to estimate c by yourself. Once you figured you have got a good answer, you can Google the keyword “Basel Problem”.

3.13. Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1, 2, \ldots$.
(a) Estimate the value of c numerically. Note that the series converges.
(b) Find $P[X > 4]$.
(c) Find $P[6 \le X \le 8]$.

Question 50: [Basic] Problem 3.17.

3.17. A modem transmits a +2 voltage signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{4/10, 3/10, 2/10, 1/10\}$.
(a) Find the pmf of the output Y of the channel.
(b) What is the probability that the output of the channel is equal to the input of the channel?
(c) What is the probability that the output of the channel is positive?

3.18 A computer reserves a path in a network

Question 51: [Basic] Problem 3.20.

makes the wrong decision.

3.20. Two dice are tossed and we let X be the difference in the number of dots facing up.
(a) Find and plot the pmf of X.
(b) Find the probability that $|X| \le k$ for all k.

Question 52: [Basic] Problem 3.28.

3.28. Find the expected value and variance of the modem signal in Problem 3.17.

Question 53: [Basic] Write down the sample space and weight assignment (pmf) for a binomial random variable X with parameters $n = 4$, $p = 1/4$. Plot the pmf of X. Compute its mean $E(X)$ and variance $\text{Var}(X)$.

Question 54: [Basic] Write down the sample space and weight assignment (pmf) for a geometric random variable X with parameters $p = 1/3$. Compute its mean $E(X)$ and the probability $P(X \leq 5 \text{ and } X \text{ is a prime number})$.

Question 55: [Basic] Write down the sample space and weight assignment (pmf) for a Poisson random variable X with parameters $\alpha = 2$. Compute its mean $E(X)$ and the probability variance $P(X \leq 5 \text{ and } X \text{ is a prime number})$.

Question 56: [Basic] This question demonstrates that we can easily convert one random experiment to a random variable.

Carlos tosses two fair coins simultaneously and let Z denote the outcome. Namely, the possible outcomes of $Z = (Z_1, Z_2)$ are $S_Z = \{(0,0),(0,1),(1,0),(1,1)\}$ where Z_1 and Z_2 are the outcomes of the first and the second coins respectively.

Let $f(a,b)$ be a function that counts the number of 1s (heads). Namely,

$$f(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ 1 & \text{if } a = 0, b = 1 \text{ or } a = 1, b = 0 \\ 2 & \text{if } a = b = 1 \end{cases}. \tag{1}$$

Let $Y = f(Z_1, Z_2)$. What is the sample space of Y? What is the pmf of Y? Is Y a binomial random variable? If yes, what are the values of the parameters n and p?

Question 57: [Intermediate/Exam Level] Continue from the last question.

Michael also tosses two fair coins and let $W = (W_1, W_2)$ denote the corresponding outcomes. What is the sample space when we consider jointly Z and W? What is the corresponding weight assignment? (Hint 1: $S = \{(0,0,0,0),(0,0,0,1),\cdots\}$. Hint 2: You need to assume independence between Z and W.)

Let $X = \max(f(Z_1,Z_2), f(W_1,W_2))$. What is the sample space of X? What is the pmf of X?

(Optional: You should compare these two questions with Problems 3.1 and 3.11.)

3.1. Let X be the maximum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

3.11. Let X be the maximum of the coin tosses in Problem 3.1.

(a) Compare the pmf of X with the pmf of Y, the number of heads in two tosses of a fair coin. Explain the difference.

(b) Suppose that Carlos uses a coin with probability of heads $p = 3/4$. Find the pmf of X.

Question 58: [Intermediate/Exam Level] A slight modification of Problem 3.52.

A sequence of bits is transmitted over a channel that introduces errors (to each bit independently) with bit-by-bit error probability $p = 0.01$.

1. What is the pmf of N, the number of error-free bits before the first erroneous bit?
2. What is $E(N)$?
3. What is the probability $P(N \geq 1000)$? Suppose we can use some signal processing technique to reduce the value of the bit-by-bit error probability p. What is the appropriate value of p such that $P(N \geq 1000) > 0.99$. Namely, we want to be 99% sure that there will be 1000 consecutive error-free bits before the occurrence of the first erroneous bit.

Question 59: [Intermediate/Exam Level] A modification of Problem 3.54.

Let M be a geometric random variable with parameter p. Show that M satisfies

$$P(M \geq k + j | M \geq j) = P(M \geq k). \tag{2}$$

(Optional: This property is called "the memoryless property." Can you explain why this is called the *memoryless* property?)

Question 60: [Advanced] Problem 3.59. In addition to Problem 3.59, also answer the following question: The web server crashes if there are more than 15 packets arriving in a 100ms interval. What is the probability that a web-server crashes in the duration from 0-th ms to 100-th ms? (Hint: you should write down the expression first. You then need a calculator/Matlab to compute this probability.)

(In practice, the web-traffic is widely modeled by a Poisson distribution. Many network engineers use a similar procedure to estimate the reliability of a server.)

3.59. The number of page requests that arrive at a Web server is a Poisson random variable with an average of 6000 requests per minute.

(a) Find the probability that there are no requests in a 100-ms period.

(b) Find the probability that there are between 5 and 10 requests in a 100-ms period.

Question 61: [Intermediate/Exam Level] Problem 3.65.

3.65. An LCD display has 1000×750 pixels. A display is accepted if it has 15 or fewer faulty pixels. The probability that a pixel is faulty coming out of the production line is 10^{-5}. Find the proportion of displays that are accepted.