ECE 302-003 Homework #4 Solution

Fall 2023

Question 38:

a: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

b: $P(HHH) = P(HHT) = \cdots = P(TTT) = \frac{1}{8}$ (all outcomes are equally likely)

$P(X=0) = P(TTT) = \frac{1}{8}$

$P(X=1) = P(HTT) + P(THT) + P(TTH) = \frac{3}{8}$

$P(X=2) = P(HHT) + P(HTH) + P(THH) = \frac{3}{8}$

$P(X=3) = P(HHH) = \frac{1}{8}$

c: $S_X = \{0, 1, 2, 3\}$

d:
$$P_X(x) = \begin{cases} \frac{1}{8} & x = 0, 3 \\ \frac{3}{8} & x = 1, 2 \\ 0 & \text{o/w} \end{cases}$$

e: $S_Y = \{0, 1, 2, 3\}$ $\quad P_Y(y) = P_X(x)$

f: $S_{X,Y} = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3),$
$(2,0), (2,1), (2,2), (2,3), (3,0), (3,1), (3,2), (3,3)\}$

g: $\cancel{P((0,0)) = P((0,1)) = \cdots = P((3,3))}$

$P((0,0)) = \frac{1}{64} = P((3,3)) = P((0,3)) = P((3,0))$

$P((0,1)) = P((0,2)) = P((1,0)) = P((1,3)) = P((2,0)) = P((2,3)) = P((3,1)) = P((3,2)) = \frac{3}{64}$

$P((1,1)) = P((1,2)) = P((2,1)) = P((2,2)) = \frac{9}{64}$

f: $P((0,0)) + P((1,1)) + P((2,2)) + P((3,3)) = \frac{1}{64} + \frac{9}{64} + \frac{9}{64} + \frac{1}{64} = \frac{20}{64} = \frac{5}{16}$

Question 39:

a: $S = \{ (0,0), (0,1), (1,0), (1,1) \}$

b: $P((0,0)) = P((0,1)) = P((1,0)) = P((1,1)) = \frac{1}{4}$

c: $P(A) = P(X=1) = \frac{1}{2}$

$P(B) = P(Y=1) = \frac{1}{2}$

$P(C) = P(M=1) = \frac{1}{2}$

$P(A \cap B) = P(X=1 \,\&\, Y=1) = P((1,1)) = \frac{1}{4}$

$P(A \cap C) = P(X=1 \,\&\, M=1) = P((1,0)) = \frac{1}{4}$

$P(B \cap C) = P(Y=1 \,\&\, M=1) = P((0,1)) = \frac{1}{4}$

$P(A \cap B) = P(A)P(B) \Rightarrow$ A & B are independent

$P(A \cap C) = P(A)P(C) \Rightarrow$ A & C are independent

$P(B \cap C) = P(B)P(C) \Rightarrow$ B & C are independent

d: $P(A \cap B \cap C) = P(X=1 \,\&\, Y=1 \,\&\, M=1) = 0$

$P(A \cap B \cap C) \neq P(A)P(B)P(C) \Rightarrow (A, B, C)$ are not jointly independent

Question 40:

1) a: $\int_0^{2\pi} a\cos(\omega t+\theta)\,d\theta$

$= a\sin(\omega t+\theta)\Big|_0^{2\pi}$

$= a\left[\sin(\omega t+2\pi) - \sin(\omega t)\right]$

$= 0$

b: $\int_0^{2\pi} a\cos(\omega t+\theta)\,da$

$= \cos(\omega t+\theta)\left[\frac{1}{2}a^2\right]_0^{2\pi}$

$= \cos(\omega t+\theta)\left(\frac{1}{2}(4\pi^2) - 0\right)$

$= 2\pi^2\cos(\omega t+\theta)$

Question 41:

$$f_X(x) = \begin{cases} x & 0 \le x \le 1 \\ \frac{1}{2} & 1 < x \le 2 \\ 0 & \text{ow} \end{cases} \qquad F_X(x) = \int_{-\infty}^{x} f_X(s)\,ds$$

a: for $x < 0$, $F_X(x) = 0$

b: for $0 \le x \le 1$, $F_X(x) = \int_0^x s\,ds = \frac{1}{2} s^2 \Big|_0^x = \frac{1}{2} x^2$

c: for $1 < x \le 2$, $F_X(x) = \frac{1}{2} + \int_1^x \frac{1}{2}\,ds = \frac{1}{2} + \frac{1}{2} s \Big|_1^x = \frac{1}{2} + \frac{1}{2}(x-1) = \frac{1}{2} x$

d: for $x > 2$; $F_X(x) = 1$

e:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} x^2 & 0 \le x \le 1 \\ \frac{1}{2} x & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$$

Question 42:

$$f(x) = \frac{3}{2} e^{-3|x|}$$

1. $F(x) = \int_{-\infty}^{x} \frac{3}{2} e^{-3(\cancel{-3s})} ds \quad \text{for} \quad x < 0$

$$= \int_{-\infty}^{x} \frac{3}{2} e^{3s} ds = \frac{1}{2} e^{3s} \Big|_{-\infty}^{x} = \frac{1}{2}(e^{3x} - 0) = \frac{1}{2} e^{3x}$$

2. $F(x) = \int_{-\infty}^{0} \frac{3}{2} e^{3s} ds + \int_{0}^{x} \frac{3}{2} e^{-3s} ds \quad \text{for} \quad x > 0$

$$= \frac{1}{2} e^{3s} \Big|_{-\infty}^{0} - \frac{1}{2} e^{-3s} \Big|_{0}^{x} = \frac{1}{2}(1 - 0) - \frac{1}{2}(e^{-3x} - 1)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} e^{-3x} = 1 - \frac{1}{2} e^{-3x}$$

3. $\lim_{x \to \infty} F(x) = 1$

4. $P_k = \int_{-(k+1)}^{-k} f(x) dx + \int_{k}^{k+1} f(x) dx \quad k \text{ integer} \geq 0$

$$= \int_{-(k+1)}^{-k} \frac{3}{2} e^{3x} dx + \int_{k}^{k+1} \frac{3}{2} e^{-3x} dx$$

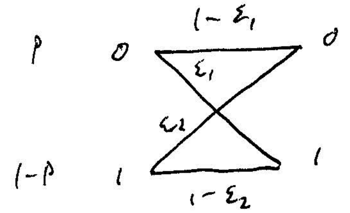
$$= \frac{1}{2} e^{3x} \Big|_{-(k+1)}^{-k} - \frac{1}{2} e^{-3x} \Big|_{k}^{k+1}$$

$$= \frac{1}{2}\left(e^{-3k} - e^{-3(k+1)}\right) - \frac{1}{2}\left(e^{-3(k+1)} - e^{-3k}\right)$$

$$= e^{-3k}\left(\frac{1}{2} + \frac{1}{2}\right) + e^{-3(k+1)}\left(-\frac{1}{2} - \frac{1}{2}\right)$$

$$= e^{-3k} - e^{-3(k+1)}$$

Question 43:



a: $P(\text{output}=0) = P(\text{output}=0 \mid \text{input}=0)\,P(\text{input}=0) + P(\text{output}=0 \mid \text{input}=1)\,P(\text{input}=1)$

$$= ((1-\varepsilon_1)(p) + (\varepsilon_2)(1-p)$$

$$= p(1-\varepsilon_1) + \varepsilon_2(1-p)$$

b: $$P(\text{input}=0 \mid \text{output}=1) = \frac{P(\text{output}=1 \mid \text{input}=0)\,P(\text{input}=0)}{P(\text{output}=1)}$$

$$= \frac{\varepsilon_1 p}{\varepsilon_1 p + (1-\varepsilon_2)(1-p)}$$

$$P(\text{input}=1 \mid \text{output}=1) = \frac{P(\text{output}=1 \mid \text{input}=1)\,P(\text{input}=1)}{P(\text{output}=1)}$$

$$= \frac{(1-\varepsilon_2)(1-p)}{\varepsilon_1 p + (1-\varepsilon_2)(1-p)}$$

which is more probable depends on p, ε_1 and ε_2

For $p=0.5$, $\epsilon_1=0.1$, and $\epsilon_2=0.1$,

$$P(\text{input}=0 \mid \text{output}=1) = \frac{0.1\cdot 0.5}{0.1\cdot 0.5 + (1-0.1)(1-0.5)} = 0.1 \ \#$$

$$P(\text{input}=1 \mid \text{output}=1) = \frac{(1-0.1)(1-0.5)}{0.1\cdot 0.5 + (1-0.1)(1-0.5)} = 0.9 \ \#$$

For $p=0.7$, $\epsilon_1=0.3$, and $\epsilon_2=0.4$

$$P(\text{input}=0 \mid \text{output}=1) = \frac{0.3\cdot 0.7}{0.3\cdot 0.7 + (1-0.4)(1-0.7)} = \frac{7}{13} \ \#$$

$$P(\text{input}=1 \mid \text{output}=1) = \frac{(1-0.4)(1-0.7)}{0.3\cdot 0.7 + (1-0.4)(1-0.7)} = \frac{6}{13} \ \#$$

Question 44:

$P(H \mid \text{coin } A) = P_1 = \frac{1}{3}$

$P(H \mid \text{coin } B) = P_2 = \frac{2}{3}$

Note: there is 1 way to get 3 H's
there are 3 ways to get 2 H's
" " 3 " " " 1 H's
there is 1 way to get 0 H's

a: $P_3 = \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$

$= \frac{1}{2}\left(\frac{1}{27} + \frac{8}{27}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

$P_2 = \frac{1}{2}(3)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{1}{2}(3)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$

$= \frac{3}{2}\left(\frac{2}{27} + \frac{4}{27}\right) = \frac{3}{2}\left(\frac{2}{9}\right) = \frac{1}{3}$

$P_1 = \frac{1}{2}(3)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \frac{1}{2}(3)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$

$P_0 = \frac{1}{2}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$

$$P_k = \begin{cases} \frac{1}{6} & k = 0, 3 \\ \frac{1}{3} & k = 1, 2 \\ 0 & \text{ow} \end{cases}$$

b: $P(\text{coin } A \mid k=0) = \dfrac{P(k=0 \mid \text{coin } A)\, P(\text{coin } A)}{P(k=0)} = \dfrac{\left(\frac{2}{3}\right)^3 \frac{1}{2}}{\frac{1}{6}} = 3\left(\frac{8}{27}\right) = \frac{8}{9}$

$P(\text{coin } A \mid k=1) = \dfrac{P(k=1 \mid \text{coin } A)\, P(\text{coin } A)}{P(k=1)} = \dfrac{3\left(\frac{4}{9}\right)\left(\frac{1}{3}\right)\frac{1}{2}}{\frac{1}{3}} = \frac{9}{2}\left(\frac{4}{27}\right) = \frac{2}{3}$

$P(\text{coin } A \mid k=2) = \dfrac{P(k=2 \mid \text{coin } A)\, P(\text{coin } A)}{P(k=2)} = \dfrac{3\left(\frac{1}{9}\right)\left(\frac{2}{3}\right)\frac{1}{2}}{\frac{1}{3}} = \frac{9}{2}\left(\frac{2}{27}\right) = \frac{1}{3}$

$P(\text{coin } A \mid k=3) = \dfrac{P(k=3 \mid \text{coin } A)\, P(\text{coin } A)}{P(k=3)} = \dfrac{\left(\frac{1}{27}\right)\frac{1}{2}}{\frac{1}{6}} = \frac{3}{27} = \frac{1}{9}$

c: For $k=0$ & 1, coin A is more likely
For $k=2$ & 3, coin B is more likely

Question 45:

$S = \{1, 2, 3, 4\}$

$A = \{1, 2\} \quad B = \{1, 3\} \quad C = \{1, 4\} \qquad P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A \cap B) = P(1) = \frac{1}{4} = P(A)P(B)$

$P(B \cap C) = P(1) = \frac{1}{4} = P(B)P(C)$

$P(A \cap C) = P(1) = \frac{1}{4} = P(A)P(C)$

$P(A \cap B \cap C) = P(1) = \frac{1}{4} \neq P(A)P(B)P(C)$

They are not independent events

Question 46:

$U \sim \mathcal{U}(0,1)$

$A = \{0 < U < \frac{1}{2}\} \quad B = \{\frac{1}{4} < U < \frac{3}{4}\} \quad C = \{\frac{1}{2} < U < 1\}$

$P(A \cap B) = P(\frac{1}{4} < U < \frac{1}{2}) = \frac{1}{4} = P(A)P(B)$ A and B are independent

$P(A \cap C) = P(\emptyset) = 0$ A and C are NOT independent

$P(B \cap C) = P(\frac{1}{2} < U < \frac{3}{4}) = \frac{1}{4} = P(B)P(C)$ B and C are independent

$P(A \cap B \cap C) = P(\emptyset) = 0$ A, B, C are NOT independent

Question 47:

A and B independent $\Rightarrow \begin{cases} P(A \cap B) = P(A)P(B) \\ P(A|B) = P(A) \\ P(B|A) = P(B) \end{cases}$

$P(B^c|A) = 1 - P(B|A)$

$P(B^c \cap A) = P(A) - P(B \cap A) = P(A)(1 - P(B)) = P(A)P(B^c)$

Therefore, A and B^c are independent

$P(A^c|B) = 1 - P(A|B)$

$P(A^c \cap B) = P(B) - P(A \cap B) = P(B)(1 - P(A)) = P(B)P(A^c)$

Therefore, A^c and B are independent

$P(A^c/B^c) = 1 - P(A/B^c)$

$P(A^c \cap B^c) = P(B^c) - P(A|B^c)P(B^c) = P(B^c)(1 - P(A))$ since A and B^c are ind.

$= P(B^c)P(A^c)$

Therefore A^c and B^c are independent

Question 48:

a: $P(A\cup B) = P(A\cap B^c) + P(A\cap B) + P(A^c\cap B)$

$= P(A)P(B^c) + P(A)P(B) + P(A^c)P(B)$ since $A\perp B \Rightarrow \begin{cases} A^c\perp B \\ A\perp B^c \end{cases}$

$= P(A)(\underbrace{P(B) + P(B^c)}_{1}) + P(A^c)P(B)$

$= P(A) + (1-P(A))P(B) = P(B) + (1-P(B))P(A)$

b: $P(A\cup B) = P(A\cap B^c) + P(A\cap B) + P(A^c\cap B)$

$= P(A) + P(B)$

c: $P(A\cup B\cup C) = P(A\cap(B\cup C)^c) + P(A\cap(B\cup C)) + P(A^c\cap(B\cup C))$

$= P(A\cap(B^c\cap C^c)) + P((A\cap B)\cup(A\cap C)) + P((A^c\cap B)\cup(A^c\cap C))$

$= P(A)P(B^c)P(C^c) + [P((A\cap B)\cap(A\cap C)^c) + P((A\cap B)\cap(A\cap C)) + P((A\cap B)^c\cap(A\cap C))]$

$+ [P((A^c\cap B)\cap(A^c\cap C)^c) + P((A^c\cap B)\cap(A^c\cap C)) + P((A^c\cap B)^c\cap(A^c\cap C))]$

$= P(A)P(B^c)P(C^c) + [P((A\cap B)\cap(A^c\cup C^c)) + P(A\cap B\cap C) + P((A^c\cup B^c)\cap(A\cap C))]$

$+ [P((A^c\cap B)\cap(A\cup C^c)) + P(A^c\cap B\cap C) + P((A\cup B^c)\cap(A^c\cap C))]$

$= P(A)P(B^c)P(C^c) + P(A\cap B\cap C^c) + P(A\cap B\cap C) + P((A^c\cup B^c)\cap C)$

$+ P(A^c\cap B\cap C^c) + P(A^c\cap B\cap C) + P((A\cup B^c)\cap C)$

$= P(A)P(B^c)P(C^c) + P(A)P(B)P(C^c) + P(A)P(B)P(C) + P(A^c)P(B)P(C^c) + P(A^c)P(B)P(C)$

$+ P((A^c\cap C)\cup(B^c\cap C)) + P((A\cap C)\cup(B^c\cap C))$

$= P(A)P(B^c)P(C^c) + P(A)P(B)P(C^c) + P(A)P(B)P(C) + P(A^c)P(B)P(C^c) + P(A^c)P(B)P(C)$

$+ [P((A^c\cap C)\cap(B^c\cap C)^c) + P((A^c\cap C)\cap(B^c\cap C)) + P((A^c\cap C)^c\cap(B^c\cap C))]$

$+ [P((A\cap C)\cap(B^c\cap C)^c) + P((A\cap C)\cap(B^c\cap C)) + P((A\cap C)^c\cap(B^c\cap C))]$

$=$ line1 $+ [P((A^c\cap C)\cap(B\cup C^c)) + P(A^c\cap B^c\cap C) + P((A\cup C^c)\cap(B^c\cap C))]$

$+ [P((A\cap C)\cap(B\cup C^c)) + P(A\cap B^c\cap C) + P((A^c\cup C^c)\cap(B^c\cap C))]$

$= P(A)P(C^c)(P(B^c) + P(B)) + P(A^c)P(B)(P(C^c) + P(C)) + P(A)P(B)P(C)$

$+ P(A^c\cap B\cap C) + P(A^c)P(B^c)P(C) + P(A\cap B^c\cap C)$

$+ P(A\cap B\cap C) + P(A)P(B^c)P(C) + P(A^c\cap B^c\cap C)$

$$= P(A)P(C^c) + P(A^c)P(B) + P(A)P(B)P(C) + P(A^c)P(B)P(C) + P(A^c)P(B^c)P(C) + P(A)P(B^c)P(C)$$
$$+ P(A)P(B)P(C) + P(A)P(B^c)P(C) + P(A^c)P(B^c)P(C)$$
$$= P(A)P(C^c) + P(A^c)P(B) + 2P(A)P(B)P(C) + P(A^c)P(C)(P(B) + P(B^c))$$
$$\cancel{+2P(A)P(B^c)P(C) + P(C)}$$
$$+ P(B^c)P(C)(P(A) + P(A^c)) + P(A)P(C)(P(B) + P(B^c))$$
$$= P(A)P(C^c) + P(A^c)P(B) + 2P(A)P(B)P(C) + P(A^c)P(C) + P(B^c)P(C) + P(A)P(C)$$
$$= P(A) + (1-P(A))P(B) + (1-P(A))P(C) + (1-P(B))P(C) + 2P(A)P(B)P(C)$$

d: $P(A \cup B \cup C) = P(A \cap (B \cup C)^c) + P(A \cap (B \cup C)) + P(A^c \cap (B \cup C))$
$$= P(A \cap (B^c \cap C^c)) + P((A \cap B) \cup (A \cap C)) + P((A^c \cap B) \cup (A^c \cap C))$$
$$= P(A) + P(\emptyset) + P(B \cup C)$$
$$= P(A) + P(B \cap C^c) + P(B \cap C) + P(B^c \cap C)$$
$$= P(A) + P(B) + P(\emptyset) + P(C)$$
$$= P(A) + P(B) + P(C)$$