ECE 302-003, Homework #4
Due date: Wednesday 9/27/2023, 11:59pm;
Submission via Gradescope

`https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html`

Q32, Q35, Q36, Q37 of HW3 are for materials after MT1. Please try your hands on these questions if you have not already done so. There is no need to turn in your solutions of Q32, Q35, Q36, and Q37.

Question 38: [Intermediate/Exam Level] Answer the following questions in sequence.

1. One person tosses three *fair* coins. What is the sample space? Hint: $S_A = \{HHH, HHT, \cdots\}$.
2. What is the weight assignment for this experiment? Let X denote the number of heads. What is the probabilities of $X = 0$, 1, 2, 3?
3. If we treat X as a "random variable," what is the same space of X?
4. What is the weight assignment of X that is consistent with the second subquestion.
5. Consider another person who also tosses three *fair* coins *independently*. Let Y be the number of heads of his/her outcomes. What is the right sample space and weight assignment for Y.
6. What is the sample space when we consider two experiments jointly? Hint: $S = \{(0,0),(0,1),(0,2),(0,3),(1,0),\cdots\}$
7. Suppose X and Y are *independent*. What is the right weight assignment for S?
8. What is the probability that both persons have the same number of heads. Namely, what is the probability $X = Y$?

Question 39: [Intermediate/Exam Level] Flip two *independent fair* coins and denote the outcomes by X and Y for which we use 1 to stand for "head" and 0 for "tail." We then flip a magic coin. We know that a magic coin always shows "tail" if $X = Y$ and shows "head" if $X \neq Y$. Consider the following events: $A = \{X = 1\}$, $B = \{Y = 1\}$ and $C = \{M = 1\}$.

1. What is the sample space? Hint: Since the magic coin depends only on the first two coins, we can choose to leave the outcome of the magic coin out of our consideration and consider the sample space being $S = \{(0,0),(0,1),\cdots,\}$.

2. What is the weight assignment? Hint: Use the condition that X and Y are independent.

3. Show that each of the three pairs (A, B), (B, C), (C, A) are *pair-wise* independent. Hint: You need to show that $P(A \cap B) = P(A) \cdot P(B)$, $P(B \cap C) = P(B) \cdot P(C)$, $P(A \cap C) = P(A) \cdot P(C)$.

4. Show that (A, B, C) *jointly* are not independent. Hint: You need to show that $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$.

Question 40: Compute the following integrals.

$$\int_0^{2\pi} a\cos(\omega t + \theta)d\theta$$

$$\int_0^{2\pi} a\cos(\omega t + \theta)da$$

Question 41: Define a 1-D function $f_X(x)$ as follows.

$$f_X(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ \frac{1}{2} & \text{if } x \in (1, 2] \\ 0 & \text{otherwise} \end{cases}$$

Another function $F(x)$ can be defined based on the integral of $f_X(x)$ as follows:

$$F(x) = \int_{s=-\infty}^{x} f_X(s)ds.$$

1. Find the expression of $F(x)$ for the case of $x < 0$.

2. Find the expression of $F(x)$ for the case of $x \in [0, 1]$.

3. Find the expression of $F(x)$ for the case of $x \in (1, 2]$.

4. Find the expression of $F(x)$ for the case of $x > 2$.

5. Write down the complete expression of $F(x)$ by considering the above four different cases. Your answer is simply a piecewise function that considers four different cases.

Question 42: Define a 1-D function $f(x)$ as follows.

$$f(x) = 1.5e^{-3|x|}$$

Show that $\int_{-\infty}^{\infty} f(x)dx = 1$.

Let $F(x) = \int_{-\infty}^{x} f(s)ds$. Compute the following values in terms of x or k.

1. $F(x)$ if $x < 0$.
2. $F(x)$ if $x \in [0, \infty)$.
3. $\lim_{x\to\infty} F(x)$.
4. For any integer value $k \geq 0$, let $p_k = \int_{-(k+1)}^{-k} f(x)dx + \int_{k}^{k+1} f(x)dx$. Find the value of p_k.

Question 43: [Basic] Problem 2.77(a,b). Note that Figure P2.3 means that

$$P(\text{output} = 0|\text{input} = 0) = 1 - \epsilon_1 \tag{1}$$
$$P(\text{output} = 1|\text{input} = 0) = \epsilon_1 \tag{2}$$
$$P(\text{output} = 0|\text{input} = 1) = \epsilon_2 \tag{3}$$
$$P(\text{output} = 1|\text{input} = 1) = 1 - \epsilon_2 \tag{4}$$

For Problem 2.77(b), the answer depends on the values of p, ϵ_1, and ϵ_2. Please answer Problem 2.77(b) for the cases of $p = 0.5$, $\epsilon_1 = 0.1$, and $\epsilon_2 = 0.1$; and for the case of $p = 0.7$, $\epsilon_1 = 0.3$, and $\epsilon_2 = 0.4$.

2.77. A nonsymmetric binary communications channel is shown in Fig. P2.3. Assume the input is "0" with probability p and "1" with probability $1 - p$.

(a) Find the probability that the output is 0.

(b) Find the probability that the input was 0 given that the output is 1. Find the probability that the input is 1 given that the output is 1. Which input is more probable?

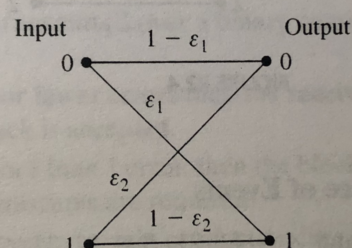

FIGURE P2.3

Question 44: [Basic] Problem 2.79(a,b,c).

2.79. One of two coins is selected at random and tossed three times. The first coin comes up heads with probability p_1 and the second coin with probability $p_2 = 2/3 > p_1 = 1/3$.

(a) What is the probability that the number of heads is k?

(b) Find the probability that coin 1 was tossed given that k heads were observed, for $k = 0, 1, 2, 3$.

(c) In part b, which coin is more probable when k heads have been observed?

(d) Generalize the solution in part b to the case where the selected coin is tossed m times. In particular, find a threshold value T such that when $k > T$ heads are observed, coin 1 is more probable, and when $k < T$ are observed, coin 2 is more probable.

(e) Suppose that $p_2 = 1$ (that is, coin 2 is two-headed) and $0 < p_1 < 1$. What is the probability that we do not determine with certainty whether the coin is 1 or 2?

Question 45: [Basic] Problem 2.82.

2.82. Let $S = \{1, 2, 3, 4\}$ and $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. Assume the outcomes are equiprobable. Are A, B, and C independent events?

Question 46: [Basic] Problem 2.83.

2.83. Let U be selected at random from the unit interval. Let $A = \{0 < U < 1/2\}$, $B = \{1/4 < U < 3/4\}$, and $C = \{1/2 < U < 1\}$. Are any of these events independent?

Question 47: [Intermediate] Problem 2.85.

2.85. Show that if A and B are independent events, then the pairs A and B^c, A^c and B, and A^c and B^c are also independent.

Question 48: [Intermediate] Problem 2.87.

2.86. Show that events A and B are

2.87. Let A, B, and C be events with probabilities $P[A]$, $P[B]$, and $P[C]$.

(a) Find $P[A \cup B]$ if A and B are independent.

(b) Find $P[A \cup B]$ if A and B are mutually exclusive.

(c) Find $P[A \cup B \cup C]$ if A, B, and C are independent.

(d) Find $P[A \cup B \cup C]$ if A, B, and C are pairwise mutually exclusive.