ECE 302-003 Homework #3 Solution

Fall 2023

Question 23:

$p(c,d) = \frac{3}{8}$

$p(b,c) = \frac{6}{8}$

$p(d) = \frac{1}{8}$

$p(c) = p(c,d) - p(d) = \frac{3}{8} - \frac{1}{8} = \frac{2}{8}$

$p(b) = p(b,c) - p(c) = \frac{6}{8} - \frac{2}{8} = \frac{4}{8}$

$p(a) = 1 - p(b) - p(c) - p(d) = 1 - \frac{4}{8} - \frac{2}{8} - \frac{1}{8} = \frac{1}{8}$

$p(a) = \frac{1}{8}$

$p(b) = \frac{4}{8}$

$p(c) = \frac{2}{8}$

$p(d) = \frac{1}{8}$

Question 24:

a: $P[A] - P[A \cap B]$

$P[B] - P[A \cap B]$

b: $\left(P[A] - P[A \cap B]\right) + \left(P[B] - P[A \cap B]\right) = P[A \text{ only}] + P[B \text{ only}]$

$= P[A] + P[B] - 2P[A \cap B]$

c: $1 - P[A] - P[B] + P[A \cap B]$

Question 25:

a: $P(X \in (-\infty, s]) = P(X \in (-\infty, r]) + P(X \in (r, s])$

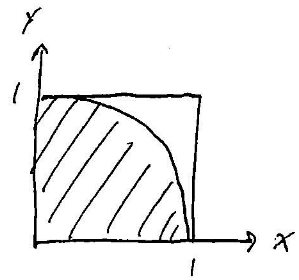
and $P(X \in (r, s]) \geq 0$

So, $P(X \in (-\infty, s]) \geq P(X \in (-\infty, r])$

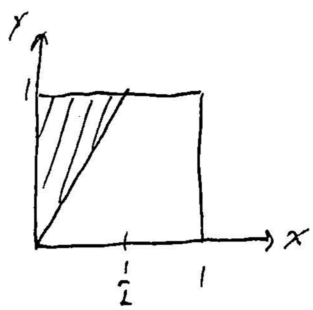
b: $P(X \in (r, s]) = P(X \in (-\infty, s]) - P(X \in (-\infty, r])$

Question 26:

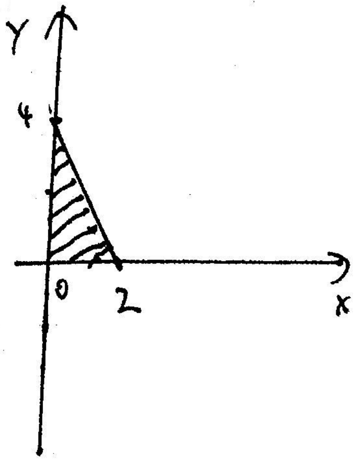
a: $P(X^2 + Y^2 \leq 1) = \dfrac{\text{Area inside the circle}}{\text{Area of the square}} = \dfrac{\frac{\pi}{4}}{1} = \dfrac{\pi}{4}$



b: $P(Y > 2X) = \dfrac{\text{Area inside the triangle}}{\text{Area of the square}} = \dfrac{\frac{1}{2}(\frac{1}{2})(1)}{1} = \dfrac{1}{4}$



Question 27:



$$A = \int_0^2 \int_0^{4-2x} xy \tfrac{1}{4}\, dy\, dx$$

$$= \tfrac{1}{4}\int_0^2 x\left(\int_0^{4-2x} y\, dy\right) dx$$

$$= \tfrac{1}{4}\int_0^2 x\left(\tfrac{y^2}{2}\Big|_0^{4-2x}\right) dx$$

$$= \tfrac{1}{4}\int_0^2 x\, \tfrac{(4-2x)^2}{2}\, dx$$

$$= \tfrac{1}{2}\int_0^2 x(2-x)^2\, dx$$

$$= \tfrac{1}{2}\int_0^2 x^3-4x^2+4x\, dx$$

$$= \tfrac{1}{2}\left(\tfrac{x^4}{4}-\tfrac{4x^3}{3}+2x^2\right)\Big|_0^2$$

$$= \tfrac{1}{2}\left(4-\tfrac{32}{3}+8\right) = 6-\tfrac{16}{3} = \tfrac{2}{3}$$

$$B = \int_0^2 \int_0^{4-2x} \tfrac{1}{4} x\, dy dx = \tfrac{1}{4}\int_0^2 x\,(4-2x)\, dx = \tfrac{1}{4}\left(2x^2-\tfrac{2x^3}{3}\right)\Big|_0^2 = \tfrac{1}{4}\left(8-\tfrac{16}{3}\right) = \tfrac{2}{3}$$

$$C = \int_0^2 \int_0^{4-2x} \tfrac{1}{4} y\, dy\, dx = \tfrac{1}{4}\int_0^2 \tfrac{y^2}{2}\Big|_0^{4-2x} dx = \tfrac{1}{8}\int_0^2 (4-2x)^2\, dx$$

$$= \tfrac{1}{2}\int_0^2 x^2-4x+4\, dx = \tfrac{1}{2}\left(\tfrac{x^3}{3}+4x-2x^2\right)\Big|_0^2 = \tfrac{1}{2}\left(\tfrac{8}{3}+8-8\right) = \tfrac{4}{3}$$

$$D = \tfrac{1}{4}\int_0^2\int_0^{4-2x} x^2\, dy\, dx = \tfrac{1}{4}\int_0^2 x^2(4-2x)\, dx = \tfrac{1}{4}\left(\tfrac{4x^3}{3}-\tfrac{2x^4}{4}\right)\Big|_0^2 = \tfrac{1}{4}\left(\tfrac{32}{3}-8\right) = \tfrac{2}{3}$$

$$E = \tfrac{1}{4}\int_0^2\int_0^{4-2x} y^2\, dy\, dx = \tfrac{1}{4}\int_0^2 \tfrac{(4-2x)^3}{3}\, dx = \tfrac{2}{3}\int_0^2 -x^3+6x^2-12x+8\, dx$$

$$= \tfrac{2}{3}\left(-\tfrac{x^4}{4}+2x^3-6x^2+8x\right)\Big|_0^2 = \tfrac{2}{3}(-4+16-24+16) = \tfrac{8}{3}$$

Question 28:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^{\infty}\int_0^{x} c e^{-x} e^{-y}\,dy\,dx$$

$$= c\int_0^{\infty} e^{-x}\left(-e^{-y}\right)\Big|_0^{x}\,dx = c\int_0^{\infty} e^{-x}\left(1-e^{-x}\right)dx = c\int_0^{\infty}\left(e^{-x}-e^{-2x}\right)dx$$

$$= c\left[-e^{-x}+\frac{1}{2}e^{-2x}\right]_0^{\infty} = c\left[-(0-1)+\frac{1}{2}(0-1)\right] = c\left(1-\frac{1}{2}\right) = \frac{1}{2}c$$

$$\frac{1}{2}c = 1 \quad \Rightarrow \quad c = 2$$

Question 29:

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{ow} \end{cases} \qquad f_Y(y) = \frac{1}{2} e^{-|y|}$$

a: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_X(x) f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{0}^{\infty} (x+y) e^{-x} \tfrac{1}{2} e^{-|y|}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\tfrac{1}{2} e^{-|y|} x e^{-x} + \tfrac{1}{2} y e^{-|y|} e^{-x}\right) dx\,dy \qquad \begin{matrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{matrix}$$

$$= \int_{-\infty}^{\infty} \left[\tfrac{1}{2} e^{-|y|}\left(-xe^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}dx\right) + \tfrac{1}{2} y e^{-|y|}\left(-e^{-x}\right)\Big|_0^{\infty}\right] dy$$

$$= \int_{-\infty}^{\infty} \left[\tfrac{1}{2} e^{-|y|}\left((0-0) - e^{-x}\Big|_0^{\infty}\right) + \tfrac{1}{2} y e^{-|y|}(0+1)\right] dy$$

$$= \int_{-\infty}^{\infty} \left[\tfrac{1}{2} e^{-|y|} + \tfrac{1}{2} y e^{-|y|}\right] dy = \int_{-\infty}^{\infty} \tfrac{1}{2}(y+1) e^{-|y|}\,dy$$

$$= \int_{-\infty}^{0} \tfrac{1}{2}(y+1)e^{y}\,dy + \int_{0}^{\infty} \tfrac{1}{2}(y+1)e^{-y}\,dy \qquad \begin{matrix} u = \frac{1}{2}(y+1) & dv = e^{y}dy & dv = e^{-y}dy \\ du = \frac{1}{2}dy & v = e^{y} & v = -e^{-y} \end{matrix}$$

$$= \tfrac{1}{2}(y+1)e^{y}\Big|_{-\infty}^{0} - \tfrac{1}{2}\int_{-\infty}^{0} e^{y}dy - \tfrac{1}{2}(y+1)e^{-y}\Big|_0^{\infty} + \tfrac{1}{2}\int_0^{\infty} e^{-y}dy$$

$$= \tfrac{1}{2}(1-0) - \tfrac{1}{2}e^{y}\Big|_{-\infty}^{0} - \tfrac{1}{2}(0-1) - \tfrac{1}{2}e^{-y}\Big|_0^{\infty}$$

$$= \tfrac{1}{2} - \tfrac{1}{2}(1-0) + \tfrac{1}{2} - \tfrac{1}{2}(0-1) = \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1$$

b: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x+y)}{10}} f_X(x) f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-\frac{(x+y)}{10}} e^{-x} \tfrac{1}{2} e^{-|y|}\,dx\,dy$$

$$= \left[\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-\frac{y}{10}} e^{-|y|}dy\right]\left[\int_0^{\infty} e^{-\frac{x}{10}} e^{-x}dx\right]$$

$$\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-\frac{y}{10}} e^{-|y|}dy = \tfrac{1}{2}\left[\int_{-\infty}^{0} e^{-\frac{y}{10}} e^{y}dy + \int_0^{\infty} e^{-\frac{y}{10}} e^{-y}dy\right]$$

$$= \tfrac{1}{2}\left[\int_{-\infty}^{0} e^{\frac{9}{10}y}dy + \int_0^{\infty} e^{-\frac{11}{10}y}dy\right] = \tfrac{1}{2}\left[\tfrac{10}{9}e^{\frac{9}{10}y}\Big|_{-\infty}^{0} - \tfrac{10}{11}e^{-\frac{11}{10}y}\Big|_0^{\infty}\right]$$

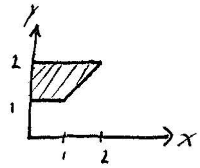
$$= \tfrac{1}{2}\left[\tfrac{10}{9}(1-0) - \tfrac{10}{11}(0-1)\right] = \tfrac{1}{2}\left(\tfrac{10}{9} + \tfrac{10}{11}\right) = \tfrac{10}{2}\left(\tfrac{1}{9} + \tfrac{1}{11}\right) = 5\left(\tfrac{11+9}{99}\right) = \tfrac{100}{99}$$

$$\int_0^{\infty} e^{-\frac{x}{10}} e^{-x}dx = \int_0^{\infty} e^{-\frac{11}{10}x}dx = -\tfrac{10}{11}e^{-\frac{11}{10}x}\Big|_0^{\infty} = \tfrac{10}{11}$$

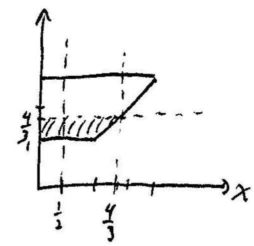
$$= \left(\tfrac{100}{99}\right)\left(\tfrac{10}{11}\right) = \tfrac{1000}{1089} \approx 0.9183$$

Question 30:

$$f(x,y) = \begin{cases} \frac{x}{y^2} & 1 \le y \le 2,\ 0 \le x \le y \\ 0 & \text{ow} \end{cases}$$



a:
$$\int_{\frac{2}{3}}^{\frac{4}{3}} \int_{\frac{1}{2}}^{\frac{3}{2}} f(x,y)\,dx\,dy$$



$$= \int_1^{\frac{4}{3}} \int_{\frac{1}{2}}^{y} \frac{x}{y^2}\,dx\,dy$$

$$= \int_1^{\frac{4}{3}} \frac{1}{y^2}\,\frac{1}{2}x^2\Big|_{\frac{1}{2}}^{y}\,dy = \int_1^{\frac{4}{3}} \frac{1}{2y^2}\left(y^2 - \frac{1}{4}\right) dy =$$

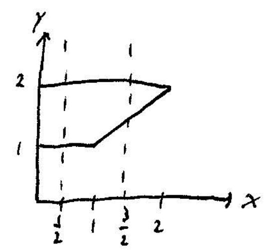
$$= \int_1^{\frac{4}{3}} \frac{1}{2}\left(1 - \frac{1}{4y^2}\right) dy = \frac{1}{2}y\Big|_1^{\frac{4}{3}} - \frac{1}{8}(-1)\left(\frac{1}{y}\right)\Big|_1^{\frac{4}{3}}$$

$$= \frac{1}{2}\left(\frac{4}{3} - 1\right) + \frac{1}{8}\left(\frac{3}{4} - 1\right) = \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{8}\left(-\frac{1}{4}\right) = \frac{1}{6} - \frac{1}{32} = \frac{16-3}{96} = \frac{13}{96}$$

b:
$$\int_{\frac{2}{3}}^{\frac{4}{3}} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_1^{\frac{4}{3}} \int_0^{y} \frac{x}{y^2}\,dx\,dy$$

$$= \int_1^{\frac{4}{3}} \frac{1}{2y^2}\,x^2\Big|_0^{y}\,dy = \int_1^{\frac{4}{3}} \frac{1}{2}\,dy = \frac{1}{2}\left(\frac{4}{3} - 1\right) = \frac{1}{6}$$

c:
$$\int_{-\infty}^{\infty} \int_{\frac{1}{2}}^{\frac{3}{2}} f(x,y)\,dx\,dy$$



$$= \int_1^{\frac{3}{2}} \int_{\frac{1}{2}}^{y} \frac{x}{y^2}\,dx\,dy + \int_{\frac{3}{2}}^{2} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{x}{y^2}\,dx\,dy$$

$$= \int_1^{\frac{3}{2}} \frac{1}{2y^2}\,x^2\Big|_{\frac{1}{2}}^{y}\,dy + \int_{\frac{3}{2}}^{2} \frac{1}{2y^2}\,x^2\Big|_{\frac{1}{2}}^{\frac{3}{2}}\,dy$$

$$= \int_1^{\frac{3}{2}} \frac{1}{2y^2}\left(y^2 - \frac{1}{4}\right) dy + \int_{\frac{3}{2}}^{2} \frac{1}{2y^2}\left(\frac{9}{4} - \frac{1}{4}\right) dy$$

$$= \int_1^{\frac{3}{2}} \frac{1}{2}\left(1 - \frac{1}{4y^2}\right) dy + \int_{\frac{3}{2}}^{2} \frac{1}{y^2}\,dy = \frac{1}{2}y\Big|_1^{\frac{3}{2}} - \frac{1}{8}\left(-\frac{1}{y}\right)\Big|_1^{\frac{3}{2}} - \frac{1}{y}\Big|_{\frac{3}{2}}^{2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{8}\left(\frac{2}{3} - 1\right) - \left(\frac{1}{2} - \frac{2}{3}\right) = \frac{1}{4} + \frac{1}{8}\left(\frac{-1}{3}\right) - \left(\frac{3-4}{6}\right)$$

$$= \frac{1}{4} - \frac{1}{24} + \frac{1}{6} = \frac{6-1+4}{24} = \frac{9}{24} = \frac{3}{8}$$

Question 31:

Let $X =$ "number on first roll" & $Y =$ "number on second roll"

$A = \{X \geq Y\}$ $\quad B = \{X = 6\}$

a: $P(A|B) = P(X \geq Y \mid X = 6) = \dfrac{P(X \geq Y \;\&\; X = 6)}{P(X=6)}$

$$= \frac{P(Y \leq 6 \;\&\; X = 6)}{P(X=6)} = \frac{P(X=6)}{P(X=6)} = \underline{1}$$

b: $P(B|A) = P(X = 6 \mid X \geq Y) = \dfrac{P(X = 6 \;\&\; Y \leq 6)}{P(X \geq Y)} = \dfrac{P(X=6)}{P(X \geq Y)} = \dfrac{\frac{1}{6}}{P(X \geq Y)}$

$$\begin{aligned} P(X \geq Y) = \; & P((1,1)) + P((2,1)) + P((3,1)) + P((4,1)) + P((5,1)) + P((6,1)) \\ & + P((2,2)) + P((3,2)) + P((4,2)) + P((5,2)) + P((6,2)) \\ & + P((3,3)) + P((4,3)) + P((5,3)) + P((6,3)) \\ & + P((4,4)) + P((5,4)) + P((6,4)) \\ & + P((5,5)) + P((6,5)) \\ & + P((6,6)) \\ = \; & \frac{1}{36}(6+5+4+3+2+1) = \frac{21}{36} \end{aligned}$$

$$P(B|A) = \frac{\frac{1}{6}}{\frac{21}{36}} = \frac{6}{21} = \frac{2}{7}$$

Question 32:

$P(A \text{ wins} \mid B \text{ is ahead}) = 0.7$

$P(A \text{ wins} \mid A \text{ and } B \text{ are tied}) = 0.5$

$P(A \text{ wins} \mid A \text{ is ahead}) = 0.4$

a: $P(AA) = (0.5)(0.4) = 0.2$

$P(ABA) = (0.5)(0.6)(0.5) = 0.15$

$P(ABB) = (0.5)(0.6)(0.5) = 0.15$

$P(BAA) = (0.5)(0.7)(0.5) = 0.175$

$P(BAB) = (0.5)(0.7)(0.5) = 0.175$

$P(BB) = (0.5)(0.3) = 0.15$

b: $P(A \text{ wins}) = P(AA) + P(ABA) + P(BAA) = 0.525$

c: $P(A \text{ wins the series} \mid B \text{ wins the first game}) = \dfrac{P(BAA)}{P(B \text{ wins the first game})} = \dfrac{0.175}{0.5} = 0.35$

d: $P(A \text{ wins the series} \mid A \text{ wins the first game}) = \dfrac{P(AA) + P(ABA)}{P(A \text{ wins the first game})} = \dfrac{0.35}{0.5} = 0.7$

e: $P(\text{one team wins 2 games while the other wins 1 game} \mid A \text{ wins the series})$

$$= \frac{P(ABA) + P(BAA)}{P(A \text{ wins})} = \frac{0.325}{0.525} = 0.619$$

Question 33:

$X \in [-1, 2]$

$A = \{X < 0\}$, $B = \{ |x - \frac{1}{2}| < \frac{1}{2}\}$, $C = \{X > \frac{3}{4}\}$

a: $P(A|B) = P(X<0 \mid |X-\frac{1}{2}| < \frac{1}{2}\} = \dfrac{P(X<0 \;\&\; |X-\frac{1}{2}|<\frac{1}{2}\}}{P(|X-\frac{1}{2}|<\frac{1}{2}\}} = \dfrac{P(X<0 \;\&\; 0<X<1)}{P(0<X<1)} = 0$

b: $P(B|C) = P(0<X<1 \mid X > \frac{3}{4}) = \dfrac{P(\frac{3}{4} < X < 1)}{P(X \geq \frac{3}{4})} = \dfrac{\frac{1}{12}}{\frac{1}{12}+\frac{1}{3}} = \dfrac{\frac{1}{12}}{\frac{5}{12}} = \frac{1}{5}$

c: $P(A|C^c) = P(X<0 \mid X<\frac{3}{4}) = \dfrac{P(X<0)}{P(X<\frac{3}{4})} = \dfrac{\frac{1}{3}}{\frac{1}{3}+\frac{3}{4}(\frac{1}{3})} = \dfrac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}} = \dfrac{\frac{1}{3}}{\frac{7}{12}} = \dfrac{4}{7}$

d: $P(B|C^c) = P(0<X<1 \mid X<\frac{3}{4}) = \dfrac{P(0<X<\frac{3}{4})}{P(X<\frac{3}{4})} = \dfrac{\frac{3}{4}(\frac{1}{3})}{\frac{3}{4}(\frac{1}{3})+\frac{1}{3}} = \dfrac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{3}} = \dfrac{\frac{1}{4}}{\frac{7}{12}} = \dfrac{3}{7}$

Question 34:

$A \cap B = \emptyset$: $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = 0$

$A \subset B$: $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(A)}{P(B)}$

$B \subset A$: $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

Question 35:

a: $M=1$: $P(\text{defective}) = k$

$M=2$: $P(\text{defective}) = 1 - P(\text{both are not defective})$

$= 1 - (1-k)(1-k)$

$= 1 - (1-k)^2$

b: $M=1$: $P(\text{defective}) = \frac{1}{2}$

$M=2$: $P(\text{defective}) = 1 - (1-\frac{1}{2})^2$

$= 1 - \frac{1}{4} = \frac{3}{4}$

c: $P(\text{defective}) = 1 - (\frac{1}{2})^M$

$1 - (\frac{1}{2})^M \geq 0.99$

$0.01 \geq (\frac{1}{2})^M$

$2^M \geq 100$ take log of both sides

$M \ln 2 \geq \ln 100$

$M \geq \frac{\ln 100}{\ln 2} = 6.6438$

$M \geq 7$

Question 36:

a: $S = \{(A,0), (A,1), (B,0), (B,1), (C,0), (C,1)\}$

where $(A,0)$ means A was chosen and was not defective

b: $P((A,0)) = P(A \text{ wasn't defective} \mid A \text{ was chosen})\, P(A \text{ was chosen}) = (0.995)(\tfrac{1}{2}) = 0.4975$

$P((A,1)) = (0.005)(\tfrac{1}{2}) = 0.0025$

$P((B,0)) = (0.999)(\tfrac{1}{10}) = 0.0999$

$P((B,1)) = (0.001)(\tfrac{1}{10}) = 0.0001$

$P((C,0)) = (0.99)(0.4) = 0.396$

$P((C,1)) = (0.01)(0.4) = 0.004$

c: $P(A \text{ was chosen} \mid \text{defective})$

$P(C \text{ was chosen} \mid \text{defective})$

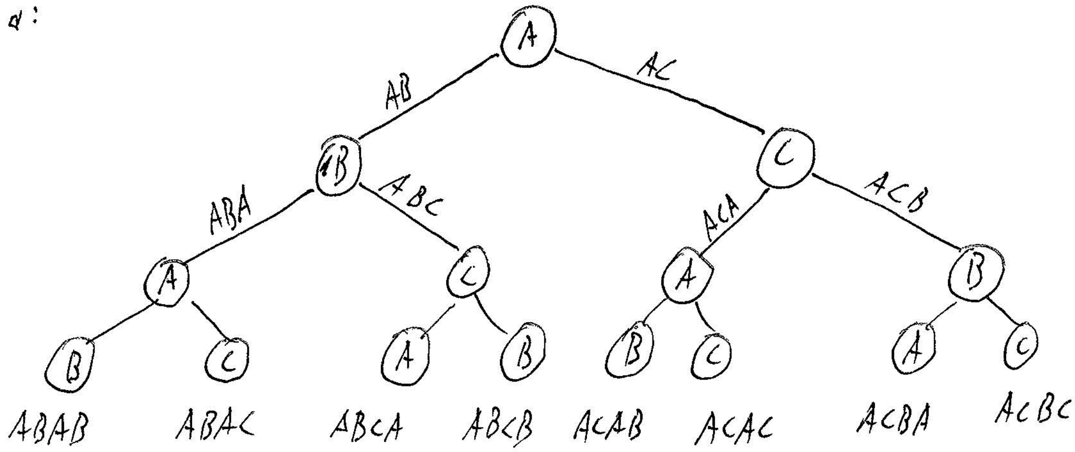
d: $P(A \text{ was chosen} \mid \text{defective}) = \dfrac{P(A \text{ chosen} \cap \text{defective})}{P(\text{defective})} = \dfrac{0.0025}{0.0025 + 0.0001 + 0.004}$

$= \dfrac{0.0025}{0.0066} = 0.3788$

$P(C \text{ was chosen} \mid \text{defective}) = \dfrac{0.004}{0.0066} = 0.6061$

Question 37:

a:



$S = \{ABAB, ABAC, ABCA, ABCB, ACAB, ACAC, ACBA, ACBC\}$

b: $P(\text{all 3 cities}) = P(ABAC) + P(ABCA) + P(ABCB) + P(ACAB) + P(ACBA) + P(ACBC)$

$= \frac{6}{8} = \frac{3}{4}$

c: $P(A \text{ twice} \mid \text{all 3 cities}) = \frac{4}{6} = \frac{2}{3}$