ECE 302-003, Homework #3
This is a self-exercise only since MT1 will be Tuesday 9/12, right before a normal Wednesday due date.
No need to submit this HW.

`https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html`

Question 23: [Basic] Textbook Problem 2.23. Here the term "elementary event" means an event that contains exactly one outcome. For example $F = \{b\}$ is an elementary event. The question is asking you to find the probability of four elementary events $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$, respectively.

2.23. A random experiment has sample space $S = \{a, b, c, d\}$. Suppose that $P[\{c, d\}] = 3/8$, $P[\{b, c\}] = 6/8$, and $P[\{d\}] = 1/8$, $P[\{c, d\}] = 3/8$. Use the axioms of probability to find the probabilities of the elementary events.

Question 24: [Basic] Textbook Problem 2.24.

2.24. Find the probabilities of the following events in terms of $P[A]$, $P[B]$, and $P[A \cap B]$:
(a) A occurs and B does not occur; B occurs and A does not occur.
(b) Exactly one of A or B occurs.
(c) Neither A nor B occur.

Question 25: [Intermediate/Exam Level] For any valid weight assignment over a continuous sample space, answer the following questions.

1. Show that we must have $P(X \in (-\infty, r]) \leq P(X \in (-\infty, s])$ if $r < s$.
2. Suppose we know the values of $P(X \in (-\infty, r])$ and $P(X \in (-\infty, s])$. What is the value of $P(X \in (r, s])$.

Question 26: [Intermediate/Exam Level] Textbook Problem 2.38. To answer this question, you first need to specify its sample space, and what kind of weight assignment you will use.

2.38. Two numbers (x, y) are selected at random from the interval $[0, 1]$.

(a) Find the probability that the pair of numbers are inside the unit circle.

(b) Find the probability that $y > 2x$.

Question 27: Define

$$f(x,y) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq x,\ 0 \leq y, \text{ and } 2x + y \leq 4 \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Find the values of

$$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dx dy \tag{2}$$

$$B = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y) dx dy \tag{3}$$

$$C = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) dx dy \tag{4}$$

$$D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f(x,y) dx dy \tag{5}$$

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f(x,y) dx dy \tag{6}$$

Question 28: Define a 2-D function $f(x, y)$ as follows.

$$f(x,y) = \begin{cases} ce^{-x}e^{-y} & \text{if } 0 \leq y \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Determine the c value such that

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x,y) dy dx = 1.$$

Question 29: Consider $f_X(x)$ and $f_Y(y)$ as follows.

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = 0.5e^{-|y|}$$

Compute the following two integrals

$$\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}(x+y)f_X(x)f_Y(y)dxdy$$

$$\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}e^{-0.1(x+y)}f_X(x)f_Y(y)dxdy$$

Question 30: Define a 2-D function $f(x,y)$ as follows.

$$f(x,y)=\begin{cases} x/y^2 & \text{if } y\in[1,2] \text{ and } x\in[0,y] \\ 0 & \text{otherwise} \end{cases}$$

Compute the values of the following 2-dimensional integrals.

$$\int_{y=2/3}^{4/3}\int_{x=1/2}^{3/2}f(x,y)dxdy$$

$$\int_{y=2/3}^{4/3}\int_{x=-\infty}^{\infty}f(x,y)dxdy$$

$$\int_{y=-\infty}^{\infty}\int_{x=1/2}^{3/2}f(x,y)dxdy.$$

Question 31: [Basic] Textbook Problem 2.62. Please assume the die is fair.

2.62. A die is tossed twice and the number of dots facing up is counted and noted in the order of occurrence. Let A be the event "number of dots in first toss is not less than number of dots in second toss," and let B be the event "number of dots in first toss is 6." Find $P[A|B]$ and $P[B|A]$.

Question 32: [Basic] Consider a best-of-three series between teams A and B. The conditional distributions are as follows.

$$P(\text{ A wins the next game} \mid \text{B is leading in the series }) = 0.7$$
$$P(\text{ A wins the next game} \mid \text{A and B are tied in the series }) = 0.5$$
$$P(\text{ A wins the next game} \mid \text{A is leading in the series }) = 0.4$$

1. Construct the weight assignment for the sample space S. Hint: use conditional distribution and similar derivation as in Example 2.25 in p. 49 attached at the end of the homework (not Problem 2.25 in p. 84).

2. $P($ A wins the series $)$?
3. $P($ A wins the series | B wins the first game$)$?
4. $P($ A wins the series | A wins the first game$)$?
5. $P($ One team wins 2 games while the other wins 1 game | A wins the series$)$?

Question 33: [Basic] Textbook Problem 2.69.

2.69. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$. Find $P[A|B]$, $P[B|C]$, $P[A|C^c]$, $P[B|C^c]$.

Question 34: [Basic] Textbook Problem 2.73(a). Express your answers in terms of $P(A)$ and $P(B)$.

2.73. **(a)** Find $P[A|B]$ if $A \cap B = \emptyset$; if $A \subset B$; if $A \supset B$.
(b) Show that if $P[A|B] > P[A]$, then $P[B|A] > P[B]$.

Question 35: [Intermediate/Exam Level] A restatement of Textbook Problem 2.76. (You should take a look of the textbook to see how other people describe a probability problem.) A factory uses the following quality control mechanism: In the end of the day, choose M products randomly from those manufactured from a given assembly line. If any one of them is defective, throw away all products from that assembly line. Answer the following questions:

1. Consider an assembly line A. If I know historically $k\%$ of the products from assembly line A are defective, what is the probability that the assembly line A is declared defective in today's inspection. Answer the above question for $M = 1$ and $M = 2$, respectively.

2. Consider an assembly line B. If I know historically 50% of the products from assembly line B are defective, what is the probability that the assembly line B is declared defective in today's inspection. Answer the above question for $M = 1$ and $M = 2$, respectively.

3. [Optional] As you can see that the larger the M value is, the higher the probability to declare "defective", namely, the more stringent the quality control is. In practice, a reputed company is very stringent in their quality control and thus chooses a large M. For assembly line B, how large should the M value be so that we declare "defective" with probability $\geq$99%.

2.76. In each lot of 100 items, two items are tested, and the lot is rejected if either of the tested items is found defective.

(a) Find the probability that a lot with k defective items is accepted.

(b) Suppose that when the production process malfunctions, 50 out of 100 items are defective. In order to identify when the process is malfunctioning, how many items should be tested so that the probability that one or more items are found defective is at least 99%?

Question 36: [Intermediate/Exam Level] Textbook Problem 2.80. Answer the following questions before trying to solve Problem 2.80:

1. What is the sample space? Hint: $S = \{(A, \text{defective}), (A, \text{not-defective}), \cdots\}$
2. What is the weight assignment/distribution on the sample space? Hint: using similar construction as in Example 2.25 in p. 49.
3. Each question in Problem 2.80 is about conditional distribution $P(E_1|E_2)$. What are the events E_1 and E_2 in each question respectively?

2.80. A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities .005, .001, and .010, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was A; that the manufacturer was C. Assume that the proportions of chips from A, B, and C are 0.5, 0.1, and 0.4, respectively.

Question 37: [Intermediate/Exam Level] A salesperson traveled between cities A, B, and C for four consecutive nights, and each night he could only stay in one city. At day 1, the salesperson started from city A and stayed there for the first night. For the next three mornings, the salesperson uniformly randomly selected his/her next destination from the other two cities (excluding the city he/she stayed for the last night). For example, if he/she stayed in city B for the third night, the fourth night of his can only be in either city A or city C.

1. What is the sample space? Hint: You can use the tree method.
2. What is the probability that the salesperson was able to visit all three cities?
3. What is the probability that the salesperson visited city A twice, given that the salesperson had visited all three cities.

In the next example we show how this equation is useful in finding probabilities in sequential experiments. The example also introduces a **tree diagram** that facilitates the calculation of probabilities.

Example 2.25

An urn contains two black balls and three white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted. Find the probability that both balls are black.

This experiment consists of a sequence of two subexperiments. We can imagine working our way down the tree shown in Fig. 2.10 from the topmost node to one of the bottom nodes: We reach node 1 in the tree if the outcome of the first draw is a black ball; then the next subexperiment consists of selecting a ball from an urn containing one black ball and three white balls. On the other hand, if the outcome of the first draw is white, then we reach node 2 in the tree and the second subexperiment consists of selecting a ball from an urn that contains two black balls and two white balls. Thus if we know which node is reached after the first draw, then we can state the probabilities of the outcome in the next subexperiment.

Let B_1 and B_2 be the events that the outcome is a black ball in the first and second draw, respectively. From Eq. (2.28b) we have

$$P[B_1 \cap B_2] = P[B_2 \mid B_1]P[B_1].$$

In terms of the tree diagram in Fig. 2.10, $P[B_1]$ is the probability of reaching node 1 and $P[B_2 \mid B_1]$ is the probability of reaching the leftmost bottom node from node 1. Now $P[B_1] = 2/5$ since the first draw is from an urn containing two black balls and three white balls; $P[B_2 \mid B_1] = 1/4$ since, given B_1, the second draw is from an urn containing one black ball and three white balls. Thus

$$P[B_1 \cap B_2] = \frac{1}{4}\frac{2}{5} = \frac{1}{10}.$$

In general, the probability of any sequence of colors is obtained by multiplying the probabilities corresponding to the node transitions in the tree in Fig. 2.10.

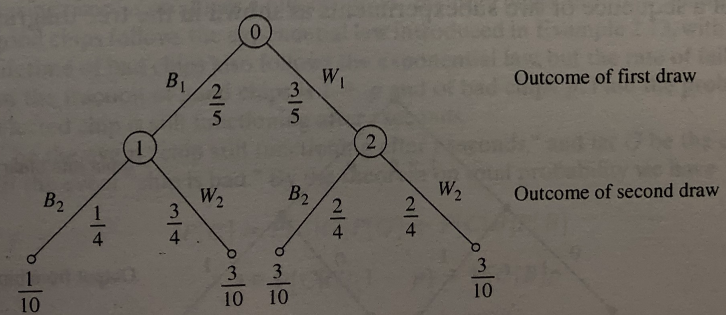

FIGURE 2.10
The paths from the top node to a bottom node correspond to the possible outcomes in the drawing of two balls from an urn without replacement. The probability of a path is the product of the probabilities in the associated transitions.