ECE 302-003 Homework #2 Solution

Fall 2023

Question 11:

a. $\Phi(w) = \int_{-\infty}^{\infty} e^{jwx} f(x)\, dx$

$$= 0.5a \int_{-\infty}^{\infty} e^{jwx - a|x|}\, dx$$

$$= 0.5a \left(\int_{-\infty}^{0} e^{jwx + ax}\, dx + \int_{0}^{\infty} e^{jwx - ax}\, dx \right)$$

Method 1

$$= 0.5a \left(\frac{1}{jw+a} e^{(jw+a)x} \Big|_{-\infty}^{0} + \frac{1}{jw-a} e^{(jw-a)x} \Big|_{0}^{\infty} \right)$$

$$= 0.5a \left(\frac{1}{jw+a} - \frac{1}{jw-a} - \lim_{x\to-\infty} \frac{1}{jw+a} e^{(jw+a)x} + \lim_{x\to\infty} \frac{1}{jw-a} e^{(jw-a)x} \right)$$

$$= 0.5a \left(\frac{1}{jw+a} - \frac{1}{jw-a} - \lim_{x\to\infty} \frac{(jw-a)e^{-(jw+a)x} - (jw+a)e^{(jw-a)x}}{(jw+a)(jw-a)} \right)$$

$$= 0.5a \left(\frac{1}{jw+a} - \frac{1}{jw-a} + \left(\lim_{x\to\infty} \frac{e^{-ax}\left(jw(e^{-jwx} - e^{jwx}) - a(e^{-jwx} + e^{jwx}) \right)}{w^2+a^2} \right) \right)$$

$$= 0.5a \left(\frac{1}{jw+a} - \frac{1}{jw-a} + \lim_{x\to\infty} \frac{e^{-ax} \to 0 \;\left(jw\cdot(-2j\cdot \sin(xw)) - a(2\cos(wx)) \right)}{w^2+a^2} \right)$$

$$= 0.5a \left(\frac{1}{jw+a} - \frac{1}{jw-a} \right) = \frac{a^2}{w^2+a^2}$$

Method 2:

swap ~~change~~ symbols:

let $x \to w$

$w \to x$

$$\Phi(w) = \int_{-\infty}^{\infty} e^{jwx} f(x)\, dx \longrightarrow \Phi(x) = \int_{-\infty}^{\infty} e^{jwx} f(w)\, dw \quad ①$$

$① \Leftrightarrow \Phi(x) = 2\pi F^{-1}\{f(w)\}$, $F^{-1}\{\}$ is inverse Fourier Transform.

By Fourier Transform Table,

$f(x) \Leftrightarrow F(w)$

$0.5a\, e^{-a|x|} \Leftrightarrow \frac{a^2}{a^2+w^2}$

By duality property,

$F(x) \Leftrightarrow 2\pi f(-w)$

So, $F(x) = \frac{a^2}{a^2+x^2} = 2\pi F^{-1}\{f(-w)\} = 2\pi F^{-1}\{0.5a\, e^{-a|-w|}\}$

$= 2\pi F^{-1}\{0.5a\, e^{-a|w|}\} = 2\pi F^{-1}\{f(w)\}$

$= \Phi(x)$

swap symbols again:

$\Phi(x) \to \Phi(w) = \frac{a^2}{a^2+w^2}$

b. $\Phi(0) = \dfrac{a^2}{a^2+0} = 1$

c. $\Phi''(\omega) = \dfrac{-2a^2(a^2-3\omega^2)}{(a^2+\omega^2)^3}$

$$\Phi''(0) = -\frac{2\cancel{}a^4}{a^6} = -\frac{2\cancel{}}{a^2}$$

Question 12:

a) $M(s) = \int_{100}^{200} e^{sx} \frac{1}{100} \, dx = \frac{1}{100s} \, e^{sx} \Big|_{100}^{200} = \frac{e^{200s} - e^{100s}}{100s}$

b) $M(0) = \frac{1-1}{0} = \frac{0}{0}$

need to use L'Hopital's rule

$\Rightarrow \left. \frac{200e^{200s} - 100e^{100s}}{100} \right|_{s=0} = \left. \left(2e^{200s} - e^{100s}\right) \right|_{s=0} = 2 - 1 = 1$

$\Rightarrow$

c) $M''(s) = \frac{d^2}{ds^2} M(s) = \frac{e^{100s}(-5000s^2 + e^{100s}(20000s^2 - 200s + 1) + 100s - 1)}{50s^3}$

$M''(0) = \frac{1(0 + 1(1) + 0 - 1)}{0} = \frac{0}{0}$

L'Hopital's rule:

$\Rightarrow M''(0) = \left. \frac{50000 \, e^{100s} (8e^{100s} - 1) s^2}{150 \, s^2} \right|_{s=0}$

again

$\Rightarrow M''(0) = \left. \frac{1000000 \, e^{100s} \, 8(-50s + e^{100s}(800s + 8) - 1)}{300 \, 8} \right|_{s=0}$

$M''(0) = \frac{1000000 \, (1(8) - 1)}{300} = \frac{7000000}{300} = \frac{70000}{3}$

Question 13:

a) $$G(z) = \sum_{k=-\infty}^{\infty} z^k p_k = \sum_{k=1}^{\infty} z^k \cdot 0.1 \cdot 0.9^k = 0.1 \cdot 0.9 z \sum_{k=1}^{\infty} (0.9z)^{k-1}$$

$$= \frac{0.09z}{1-0.9z} \quad \text{if } |0.9z| < 1 \Leftrightarrow |z| < \frac{10}{9}$$

$$= \frac{9z}{100-90z} \quad \text{if } |z| < \frac{10}{9} \quad \#$$

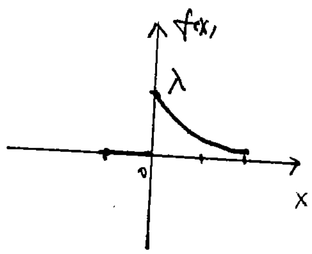
b) $$G(1) = \frac{9}{100-90} = \frac{9}{10} \quad \#$$

c) $$G'(z) = \frac{d}{dz} G(z) = \frac{9(100-90z) - 9z \cdot (-90)}{(100-90z)^2}$$

$$G'(1) = \frac{9 \cdot 10 + 9 \cdot 90}{10^2} = \frac{900}{100} = 9 \quad \#$$

Question 14:

a)



b) $$g(x,1) = \begin{cases} \dfrac{\lambda e^{-\lambda(x+1)}}{\int_1^\infty \lambda e^{-\lambda t} dt} & \text{if } x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{\lambda e^{-\lambda(x+1)}}{e^{-\lambda}} = \lambda e^{-\lambda x} & \text{if } x \geq 0 \quad = f(x) \\ 0 & \text{else} \end{cases}$$

c) $$g(x, 3.7) = \begin{cases} \dfrac{\lambda e^{-\lambda(x+3.7)}}{e^{-\lambda 3.7}} = \lambda e^{-\lambda x} & \text{if } x \geq 0 \quad = f(x) \\ 0 & \text{else} \end{cases}$$

Question 15:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\, dy\, dx$$

$$= \int_0^2\int_0^{\infty} xy \frac{1}{x} e^{-\frac{2y}{x}}\, dy\, dx = \int_0^2\int_0^{\infty} y e^{-\frac{2y}{x}}\, dy\, dx$$

$u = y \qquad dv = e^{-\frac{2y}{x}}\, dy$

$du = dy \qquad v = -\frac{x}{2} e^{-\frac{2y}{x}}$

$$= \int_0^2 \left[-y \frac{x}{2} e^{-\frac{2y}{x}} \Big|_0^{\infty} + \frac{x}{2}\int_0^{\infty} e^{-\frac{2y}{x}}\, dy \right] dx$$

$$= \int_0^2 \left[(0-0) - \frac{x^2}{4} e^{-\frac{2y}{x}} \Big|_0^{\infty} \right] dx$$

$$= \int_0^2 -\frac{x^2}{4}(0-1)\, dx = \int_0^2 \frac{1}{4} x^2\, dx$$

$$= \frac{1}{12} x^3 \Big|_0^2 = \frac{8}{12} = \frac{2}{3}$$

Question 16:

a: Sample Space = $\{(A,A), (B,B), (A,B,A), (A,B,B), (B,A,B), (B,A,A)\}$

where the letter denotes the winning team

b: If the teams are evenly matched, and each game is independent of the others, then

$P((A,A)) = P((B,B)) = \frac{1}{4}$

$P((A,B,A)) = P((A,B,B)) = P((B,A,A)) = P((B,A,B)) = \frac{1}{8}$

$2(\frac{1}{4}) + 4(\frac{1}{8}) = \frac{1}{2} + \frac{1}{2} = 1$ ✓

c: It is reasonable, but likely not accurate. Games are likely not independent, and one team is probably more likely to win than another. This is how bookies make money.

Question 17:

a: Sample Space = {0, 1, 2, 3, 4, 5}

b: A = {(4,1), (1,4), (5,2), (2,5), (6,3), (3,6)}

Question 18:

a: Sample Space = {(Al, Bob, Chris), (Al, Chris, Bob), (Bob, Al, Chris), (Bob, Chris, Al), (Chris, Al, Bob), (Chris, Bob, Al)}

b: A = {(Al, Bob, Chris), (Al, Chris, Bob)}

B = {(Al, Bob, Chris), (Chris, Bob, Al)}

C = {(Al, Bob, Chris), (Bob, Al, Chris)}

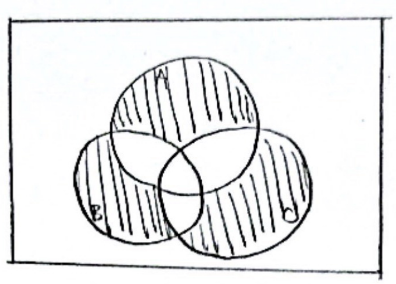
c: {(Bob, Chris, Al), (Chris, Al, Bob)}

d: {(Al, Bob, Chris)}

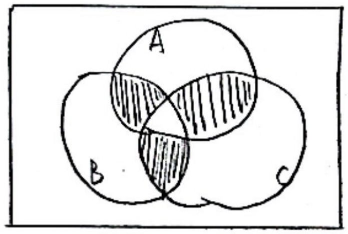
e: {(Al, Bob, Chris), (Al, Chris, Bob), (Bob, Al, Chris), (Chris, Bob, Al)}

Question 19:

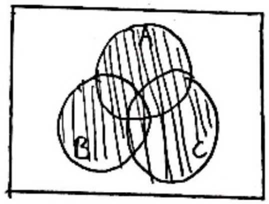
(a) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$



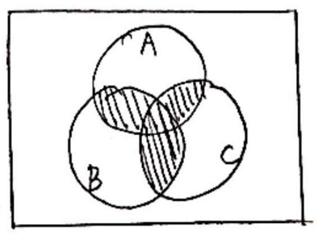
(b) $(A \cap B - A \cap B \cap C) \cup (A \cap C - A \cap B \cap C) \cup (B \cap C - A \cap B \cap C)$



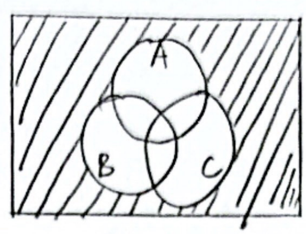
(c) $A \cup B \cup C$



(d) $(A \cap B) \cup (A \cap C - A \cap B \cap C) \cup (B \cap C - A \cap B \cap C)$



(e) $(A \cup B \cup C)^c$



Question 20:

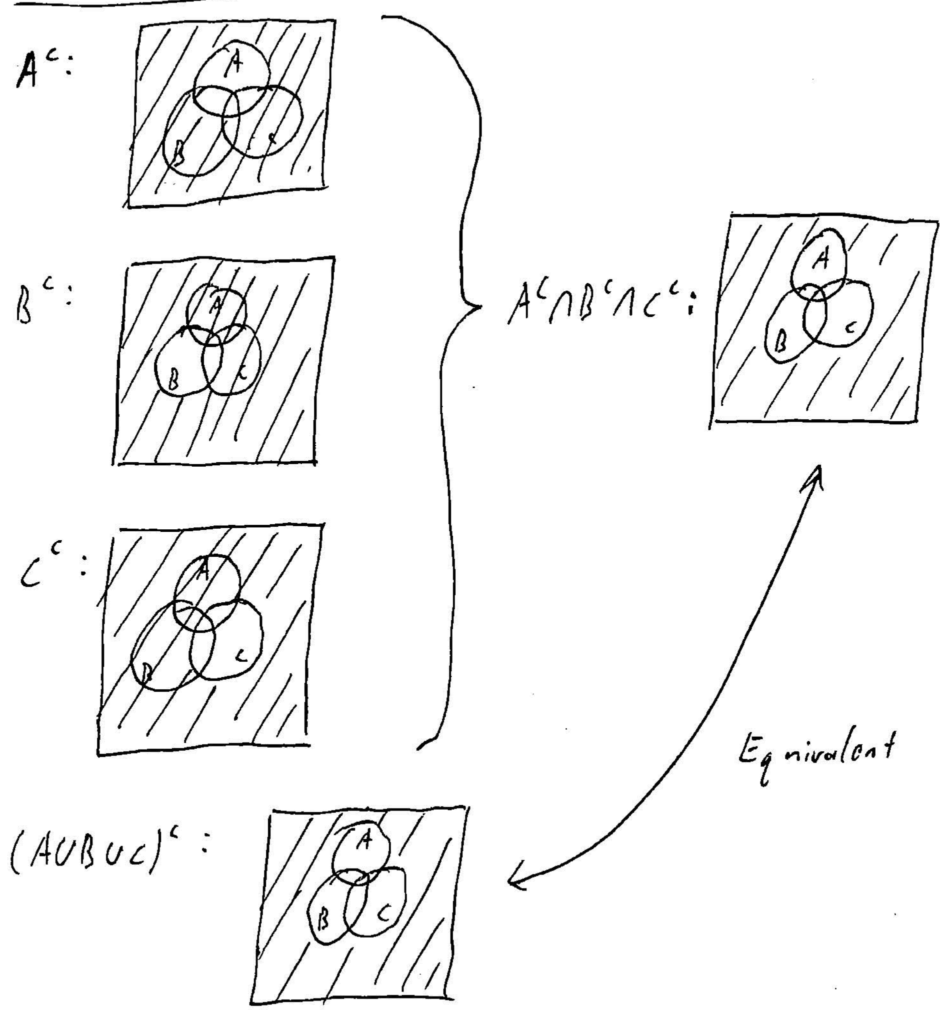
$(A \cup B \cup C)^c = (D \cup C)^c$ when $D = A \cup B$

$= D^c \cap C^c$ by DeMorgan's Rule

$= (A \cup B)^c \cap C^c$ by back-substituting

$= (A^c \cap B^c) \cap C^c$ by DeMorgan's Rule

$= A^c \cap B^c \cap C^c$



Question 21:

In [1]:
```python
import numpy as np
import numpy.random as npr

def play_game():
    # -- the game prize is hidden behind a door --
    DOORS = ["A","B","C"]
    prize = npr.choice(DOORS,1,p = [1/3, 1/3, 1/3]).item()
    # -- the player chooses a door --
    pick = npr.choice(DOORS,1,p = [1/3, 1/3, 1/3]).item()
    # -- the host reveals a non-chosen door without the prize --
    REM_NO_PRIZE = set(DOORS) - set(prize) - set(pick)
    revealed = npr.choice(list(REM_NO_PRIZE),1).item()
    # -- the player can swap or stay --
    swap = (set(DOORS) - set(pick) - set(revealed)).pop()
    # -- compute policy choices --
    return pick==prize,swap==prize

def main():
    NUM_GAMES = 10**4
    num_wins = [0,0]
    for n in range(NUM_GAMES):
        w0,w1 = play_game()
        num_wins[0] += w0
        num_wins[1] += w1
    print(num_wins)
    print("Sticking: ",num_wins[0], ", winning rate = ", num_wins[0]/NUM_GAMES)
    print("Swapping: ",num_wins[1], ", winning rate = ", num_wins[1]/NUM_GAMES)

if __name__ == "__main__":
    main()
```

```
[3312, 6688]
Sticking:  3312 , winning rate =  0.3312
Swapping:  6688 , winning rate =  0.6688
```

(c) Swapping doors yields the higher chance to win the prize.

Question 22:

In [1]:
```python
import numpy as np
import numpy.random as npr

def play_game():
    # -- the game prize is hidden behind a door --
    DOORS = ["A","B","C"]
    prize = npr.choice(DOORS,1,p = [0.5, 0.3, 0.2]).item()
    # -- the player chooses a door --
    pick = npr.choice(DOORS,1,p = [0.2, 0.4, 0.4]).item()
    # -- the host reveals a non-chosen door without the prize --
    REM_NO_PRIZE = set(DOORS) - set(prize) - set(pick)
    revealed = npr.choice(list(REM_NO_PRIZE),1).item()
    # -- the player can swap or stay --
    swap = (set(DOORS) - set(pick) - set(revealed)).pop()
    # -- compute policy choices --
    return pick==prize,swap==prize

def main():
    NUM_GAMES = 10**4
    num_wins = [0,0]
    for n in range(NUM_GAMES):
        w0,w1 = play_game()
        num_wins[0] += w0
        num_wins[1] += w1
    print(num_wins)
    print("Sticking: ",num_wins[0], ", winning rate = ", num_wins[0]/NUM_GAMES)
    print("Swapping: ",num_wins[1], ", winning rate = ", num_wins[1]/NUM_GAMES)

if __name__ == "__main__":
    main()
```

```
[2995, 7005]
Sticking:  2995 , winning rate =  0.2995
Swapping:  7005 , winning rate =  0.7005
```

(c) Swapping doors yields the higher chance to win the prize.

[Optional]:

Winning rate of "switching doors"

$= (1 - Pr(\text{prize at } A)) \cdot Pr(\text{first pick } A) + (1 - Pr(\text{prize at } B)) \cdot Pr(\text{first pick } B)$
$+ (1 - Pr(\text{prize at } C)) \cdot Pr(\text{first pick } C)$ #

Winning rate of "using the original choice"

$= Pr(\text{prize at } A) \cdot Pr(\text{first pick } A) + Pr(\text{prize at } B) \cdot Pr(\text{first pick } B)$
$+ Pr(\text{prize at } C) \cdot Pr(\text{first pick } C)$ #