ECE 302-003, Homework #2
Due date: Wednesday 9/06/2023, 11:59pm;
Submission via Gradescope

`https://engineering.purdue.edu/~chihw/23ECE302F/23ECE302F.html`

Review of Calculus:

Question 11: Define $f(x) = 0.5ae^{-a|x|}$ where $a > 0$ is a constant. And define

$$\Phi(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f(x) dx. \tag{1}$$

Sub-question 1: Find the expression of $\Phi(\omega)$.
Sub-question 2: Find the value of $\Phi(0)$.
Sub-question 3: Define the second order derivative $\Phi''(\omega) = \frac{d^2}{d\omega^2}\Phi(\omega)$. Find the value of $\Phi''(0)$.

Question 12: Define

$$f(x) = \begin{cases} \frac{1}{100} & \text{if } 100 < x < 200 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and define

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f(x) dx. \tag{3}$$

Sub-question 1: Find the expression of $M(s)$.
Sub-question 2: Find the value of $M(0)$.
Sub-question 3: Define the second order derivative $M''(s) = \frac{d^2}{ds^2}M(s)$. Find the value of $M''(0)$.

Question 13: Define

$$p_k = \begin{cases} 0.1 \cdot 0.9^k & \text{if } 0 < k \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and define

$$G(z) = \sum_{k=-\infty}^{\infty} z^k p_k. \tag{5}$$

Sub-question 1: Find the expression of $G(z)$.

Sub-question 2: Find the value of $G(1)$.

Sub-question 3: Define the first order derivative $G'(z) = \frac{d}{dz}G(z)$. Find the value of $G'(1)$.

Question 14: Define

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0 \leq x \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where λ is a positive constant.

Sub-question 1: Plot $f(x)$ for the range of $-1 < x < 2$.

Sub-question 2: For any two numbers x and y, define another function

$$g(x, y) = \begin{cases} \frac{\lambda e^{-\lambda(x+y)}}{\int_{t=y}^{\infty} \lambda e^{-\lambda t} dt} & \text{if both } 0 \leq x \text{ and } 0 \leq y \\ 0 & \text{if } x < 0 \text{ or } y < 0 \end{cases}. \tag{7}$$

Plot $g(x, 1)$ for the range of $-1 < x < 2$.

Sub-question 3: Plot $g(x, 3.7)$ for the range of $-1 < x < 2$.

Question 15: Define a 2-D function $f(x, y)$ as follows.

$$f(x, y) = \begin{cases} \frac{1}{x} e^{-\frac{2y}{x}} & \text{if } x \in (0, 2] \text{ and } y \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Compute the value of the following 2-dimensional integral.

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xyf(x, y)dydx.$$

ECE302 materials:

Question 16: [Basic] Consider a best-of-three baseball series between two teams A and B, namely, the team that wins 2 games wins the whole series. (No game will be played after one team wins the series.)

1. What is the sample space?
2. What is the weight assignment you would like to make? (You need to make sure your weight assignment is a valid one.)
3. [Optional] Is your weight assignment a reasonable one?

Question 17: [Basic] Textbook Problem 2.3 (a,b).

2.3. Two dice are tossed and the magnitude of the difference in the number of dots facing up in the two dice is noted.

(a) Find the sample space.

(b) Find the set A corresponding to the event "magnitude of difference is 3."

(c) Express each of the elementary events in this experiment as the union of elementary events from Problem 2.2.

Question 18: [Basic] Textbook Problem 2.6.

2.6. Three friends (Al, Bob, and Chris) put their names in a hat and each draws a name from the hat. (Assume Al picks first, then Bob, then Chris.)

(a) Find the sample space.

(b) Find the sets A, B, and C that correspond to the events "Al draws his name," "Bob draws his name," and "Chris draws his name."

(c) Find the set corresponding to the event, "no one draws his own name."

(d) Find the set corresponding to the event, "everyone draws his own name."

(e) Find the set corresponding to the event, "one or more draws his own name."

Question 19: [Basic] Textbook Problem 2.14. Plot the corresponding Venn diagrams.

2.14. Let A, B, and C be events. Find expressions for the following events:

(a) Exactly one of the three events occurs.

(b) Exactly two of the events occur.

(c) One or more of the events occur.

(d) Two or more of the events occur.

(e) None of the events occur.

Question 20: [Intermediate/Exam Level] Let $A, B, C \subseteq S$. Prove the following DeMorgan's Rule:

$$(A \cup B \cup C)^c = A^c \cap B^c \cap C^c.$$

Hint: (1) You can use the Venn diagram to show the above equality. (2) Or you can treat $A \cup B$ as a new event D and consider the DeMorgan's rule on $(D \cup C)^c$.

Question 21: (This question was originally created by Prof. Stanley Chan and Kent Gauen)

The Three Doors Problem is about a game show where a contestant must select a prize hidden behind one of three closed doors. (Prof. Wang discussed a simplified version under the assumption that the contestant is stubborn and always chooses door A.) The general three doors problem is now stated below.

The contestant first picks one door from three doors. Let's say they pick door A from doors A, B, and C. The host then reveals the prize is not behind one of the two remaining doors. For example, if the contestant chooses door A, then the host reveals either door B or C does not have the prize behind it. Let's say the host reveals door B. Finally, the contestant is given a choice: they can either swap doors or keep with their original guess. Since the contestant chose door A and the host revealed door B, then the contestant can either (i) swap their guess to door C or (ii) keep their guess on door A.

A youtube video describing the problem is here: `https://youtu.be/4Lb-6rxZxx0`

For this question, students need to: Write a Python script to count the number of times the contestant wins over 10,000 games when they either (i) always swap doors or (ii) always keep the same door. Based on your simulation results, answer the question which option (i.e., whether (i) or (ii)) yields the most wins if you play the game for 10,000 times?

A reference code is attached at the end of the homework. All you need to do is to complete the `main()`. You can directly use `play_game()` in your code or modify it to your liking.

When submitting your answer via Gradescope, **please (a)** provide a complete code of your `main()` and if you have modified `play_game()`, please provide the complete code for the modified `play_game()` as well.

Please (b) submit the final output of your Python code. You may type down the output of your codes, or simply screenshot the output and attach the result.

Please (c) indicate which option yields the higher chance to win the prize.

Question 22:

Continue from the previous question, we now assume the weights of the prize placement to doors A, B, and C are 0.5, 0.3, and 0.2, respectively. Namely, we use the alternative weight assignment as discussed in Prof. Wang's lecture.

The contestant has no idea where the prize is and we assume that he/she chooses doors A to C with probability 0.2, 0.4 and 0.4, respectively. Under this assumption, which option (stay vs switch) yields the most wins if we play the game many times?

For this question, students need to: Please write a short Python code and use the simulation results of 10,000 times to indicate which option yields the higher chance to win the prize. You can use your Python code of Q21 as a baseline and modify the Python for this question.

When submitting your answer via Gradescope, **please (a)** provide a complete code of your main() and if you have modified play_game(), please provide the complete code for the modified play_game() as well.

Please (b) submit the final output of your Python code. You may type down the output of your codes, or simply screenshot the output and attach the result.

Please (c) indicate which option yields the higher chance to win the prize.

Optional: Can you use simple probability-based argument to "guess" what would be the exact percentage of winning when "switching doors" and what would be the percentage of winning when "using the original choice".

(For optional question, the instruction team will provide the correct answer when posting the solution. **Students do not need to answer any optional question(s) when submitting the homework.** As long as students answer the earlier sub-questions correctly, they will receive full score for Q22.)

```python
import numpy as np
import numpy.random as npr

def play_game():
    # -- the game prize is hidden behind a door --
    DOORS = ["A","B","C"]
    prize = npr.choice(DOORS,1,p = [1/3, 1/3, 1/3]).item()
    # -- the player chooses a door --
    pick = npr.choice(DOORS,1,p = [1/3, 1/3, 1/3]).item()
    # -- the host reveals a non-chosen door without the prize --
    REM_NO_PRIZE = set(DOORS) - set(prize) - set(pick)
    revealed = npr.choice(list(REM_NO_PRIZE),1).item()
    # -- the player can swap or stay --
    swap = (set(DOORS) - set(pick) - set(revealed)).pop()
    # -- compute policy choices --
    return pick==prize,swap==prize

def main():
    # -- Please complete the main() here --

if __name__ == "__main__":
    main()
```