ECE 302-003 Homework #1 Solution

Fall 2023

Question 1:

(i) $\int_1^\infty \frac{1}{x^2}dx = \int_1^\infty x^{-2}dx = -x^{-1}\Big|_1^\infty = 0-(-1) = \boxed{1}$ #

(ii) $\int_0^\infty \lambda e^{-\lambda x}dx = -e^{-\lambda x}\Big|_0^\infty = 0-(-e^0) = 0-(-1) = \boxed{1}$ #

Question 2:

(i) Let $u = e^{-\frac{y^2}{2x^2}}$, then $\frac{du}{dy} = -\frac{2y}{2x^2}e^{-\frac{y^2}{2x^2}}$

$\Rightarrow -x^2 du = y e^{-\frac{y^2}{2x^2}}dy$

$\therefore \int y e^{-\frac{y^2}{2x^2}}dy = \int -x^2 du = -x^2\int du = -x^2 u$

$= \boxed{-x^2 e^{-\frac{y^2}{2x^2}}}$ #

(ii) $\iint x e^{yz}\,dz\,dx = \int \frac{x}{y}e^{yz}dx = \boxed{\frac{x^2}{2y}e^{yz}}$ #

Question 3:

(i) $\int_{-10}^{10} y e^{-2|y|}dy = \boxed{0}$ # $\because$ $f(y) = ye^{-2|y|}$ is an odd function and we have $\int_{-a}^{a} f(x)dx = 0$ for $a \in \mathbb{R}$ when $f(x)$ is an odd function.

(ii) Similarly, $\int_{-20}^{20} x^3 e^{-\frac{x^2}{10}}dx = \boxed{0}$ #

Question 4:

(i) $\int_0^\infty z\, 0.5\, e^{-0.5z}\, dz = \int_0^\infty \frac{1}{2}z \cdot e^{-\frac{1}{2}z}\, dz$

Let $u = \frac{1}{2}z$, $du = \frac{1}{2}dz$
$dv = e^{-\frac{1}{2}z}dz$, $v = -2e^{-\frac{1}{2}z}$

$= \left[\frac{1}{2}z \cdot (-2e^{-\frac{1}{2}z})\right]_0^\infty - \int_0^\infty \frac{1}{2} \cdot (-2e^{-\frac{1}{2}z})\, dz$

$= \left[-ze^{-\frac{1}{2}z}\right]_0^\infty + \int_0^\infty e^{-\frac{1}{2}z} dz = 0 + \left[-2e^{-\frac{1}{2}z}\right]_0^\infty$

$= 0 + (0 - (-2)) = \boxed{2}$ #

(ii) $\int_0^\infty z^2\, 0.5\, e^{-0.5z}\, dz = \int_0^\infty \frac{1}{2}z^2 \cdot e^{-\frac{1}{2}z}\, dz$

Let $u = \frac{1}{2}z^2$, $du = z\,dz$
$dv = e^{-\frac{1}{2}z}dz$, $v = -2e^{-\frac{1}{2}z}$

$= \left[\frac{1}{2}z^2 \cdot (-2)e^{-\frac{1}{2}z}\right]_0^\infty - \int_0^\infty z \cdot (-2)e^{-\frac{1}{2}z}\, dz$

$= \left[-z^2 e^{-\frac{1}{2}z}\right]_0^\infty + 2\int_0^\infty z e^{-\frac{1}{2}z}\, dz$

Note that from (i), we have
$\frac{1}{2}\int_0^\infty z e^{-\frac{1}{2}z} dz = 2$
$\therefore 2\int_0^\infty z e^{-\frac{1}{2}z} dz = 8$.

$= -(0 - 0) + 8 = \boxed{8}$ #

Question 5:

(i) $Lf(s) = \int_{-\infty}^\infty e^{-sx} f(x)dx = \int_0^\infty e^{-sx} \frac{1}{2}e^{-\frac{1}{2}x} dx = \int_0^\infty \frac{1}{2} e^{-(s+\frac{1}{2})x} dx$

$= \frac{1}{2}\frac{-1}{(s+\frac{1}{2})}\left[e^{-(s+\frac{1}{2})x}\right]_0^\infty = \boxed{\frac{1}{2}\frac{1}{(s+\frac{1}{2})} \text{ for } s > -\frac{1}{2}}$ #

(ii) $Lg(s) = \int_0^3 e^{-sx} \cdot \frac{1}{3}dx = \frac{1}{3}\left[-\frac{1}{s}e^{-sx}\right]_0^3 = \frac{1}{3}\left[-\frac{1}{s}e^{-3s} - (-\frac{1}{s})\right]$

$= \boxed{\frac{1}{3s}(1 - e^{-3s})}$ #

Question 6:

- Sub-question 1:

$$f(2, 2.01) = 0$$
$$f(1, 0.33) = \frac{2}{9} \quad \#$$

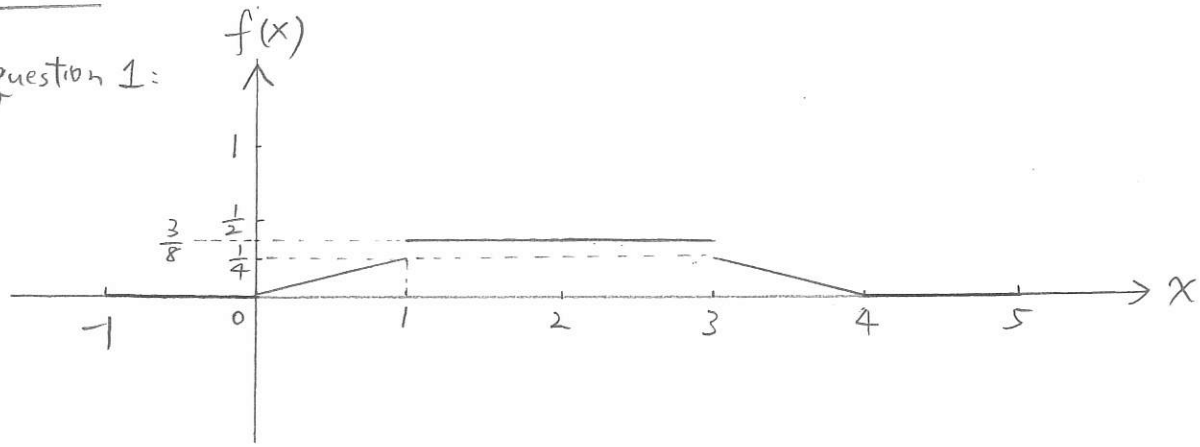
- Sub-question 2:

$$g(1,2) = \int_{t=-\infty}^{2} \int_{s=-\infty}^{1} f(s,t)\,ds\,dt = \int_{t=0}^{2} \int_{s=0}^{1} f(s,t)\,ds\,dt$$
$$= \int_{s=0}^{1} \int_{t=0}^{s} \frac{2}{9}\,dt\,ds = \int_{s=0}^{1} \left[\frac{2}{9}t\right]_{t=0}^{s} ds = \int_{s=0}^{1} \frac{2}{9}(s-0)\,ds$$
$$= \frac{2}{9}\left[\frac{1}{2}s^2\right]_{s=0}^{1} = \frac{2}{9}\cdot\frac{1}{2}(1-0)$$
$$= \boxed{\frac{1}{9}} \quad \#$$

$$g(2, 2.01) = \int_{t=0}^{2.01} \int_{s=0}^{2} f(s,t)\,ds\,dt = \int_{s=0}^{2} \int_{t=0}^{s} \frac{2}{9}\,dt\,ds$$
$$= \frac{2}{9}\left[\frac{1}{2}s^2\right]_{s=0}^{2} = \boxed{\frac{4}{9}} \quad \#$$

Question 7:

- Sub-question 1:



- Sub-question 2:

$$\int_{x=-\infty}^{\infty} x f(x)\,dx = \int_0^1 \frac{x^2}{4}\,dx + \int_1^3 \frac{3}{8}x\,dx + \int_3^4 -\frac{x^2}{4}+x\,dx$$

$$= \left[\frac{1}{12}x^3\right]_0^1 + \left[\frac{3}{16}x^2\right]_1^3 - \left[\frac{1}{12}x^3\right]_3^4 + \left[\frac{1}{2}x^2\right]_3^4$$

$$= \frac{1}{12} + \frac{24}{16} - \frac{37}{12} + \frac{7}{2}$$

$$= \boxed{2} \ \#$$

Question 8:

(i) $$\sum_{k=27}^{\infty} 0.9^k = \sum_{k'=1}^{\infty} 0.9^{k'+26} = (0.9)^{27} \cdot \sum_{k'=1}^{\infty} 0.9^{k'-1}$$

$$= (0.9)^{27} \frac{1}{1-0.9} = \boxed{10 \cdot (0.9)^{27}} \ \#$$

(ii) $$\sum_{k=0}^{\infty} k\, 0.5^{k+1} = 0 \cdot 0.5^1 + \sum_{k=1}^{\infty} k \cdot 0.5^{k+1} = 0 + 0.5^2 \cdot \sum_{k=1}^{\infty} k\, 0.5^{k-1}$$

$$= 0.5^2 \cdot \frac{1}{(1-0.5)^2} = \boxed{1} \ \#$$

(iii) $$\sum_{k=3}^{50} y^k = \sum_{k'=1}^{48} y^{k'+2} = y^3 \sum_{k'=1}^{48} y^{k'-1}$$

$$= \boxed{\begin{cases} y^3 \dfrac{(1-y^{48})}{1-y}, & \text{if } y \neq 1 \\ 48, & \text{if } y = 1 \end{cases}} \ \#$$

Question 9:

(i) $\sum_{k=-\infty}^{\infty} a_k = a_{-1} + a_9 = 0.8 + 0.2 = \boxed{1}$ #

(ii) $\sum_{k=-\infty}^{\infty} k a_k = (-1) a_{-1} + 9 \cdot a_9 = -0.8 + 1.8 = \boxed{1}$ #

(iii) $\sum_{k=-\infty}^{\infty} \min(k,3) a_k = \min(-1,3) a_{-1} + \min(9,3) a_9$

$= -1 \cdot 0.8 + 3 \cdot 0.2 = \boxed{-0.2}$ #

(iv) $\sum_{k=-\infty}^{\infty} \sin(k\pi) a_k = \sin(-\pi) \cdot 0.8 + \sin(9\pi) \cdot 0.2 = 0 + 0 = \boxed{0}$ #

(Note that $\sin(k\pi) = 0$ for $k \in \mathbb{Z}$)

Question 10:

- Sample space $S = \{ (1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (1,0), (2,0), (3,0), (4,0), (5,0), (6,0) \}$ #

- $\frac{1}{12}$

  $\because$ The die and coin are fair, i.e., head & tail is equally likely and each side of the die is equally likely.
  Besides, X and Y are independent, i.e., the outcome of X does not depend on the outcome of Y, and vice versa.
  Thus, each outcome of the sample space would have probability $\frac{1}{12}$. #

- $P(X^2 + Y \text{ is prime}) = P((1,1)) + P((2,1)) + P((4,1)) + P((6,1))$

  $= \frac{4}{12} = \boxed{\frac{1}{3}}$ #