ECE 302-003 Homework #13 Solution

Fall 2023

Question 128:

1.
$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy \quad \because X \text{ and } Y \text{ are independent.}$$
$$= \left[\int_{-\infty}^{\infty} x f_X(x)\, dx\right]\left[\int_{-\infty}^{\infty} y f_Y(y)\, dy\right]$$
$$= E(X)\,E(Y) = m_X m_Y \quad \#$$

2. $Z = X + Y$
$$E(Z) = E(X+Y) = E(X) + E(Y) = m_X + m_Y$$
$$E(Z^2) = E(X^2 + 2XY + Y^2) = E(X^2) + E(Y^2) + 2E(X)E(Y)$$
$$= Var(X) + (E(X))^2 + Var(Y) + (E(Y))^2 + 2E(X)E(Y)$$
$$= \sigma_X^2 + m_X^2 + \sigma_Y^2 + m_Y^2 + 2m_X m_Y \quad \#$$

3.
$$Var(Z) = E(Z^2) - (E(Z))^2 = \sigma_X^2 + \sigma_Y^2 + m_X^2 + m_Y^2 + 2m_X m_Y - (m_X^2 + m_Y^2 + 2m_X m_Y)$$
$$= \sigma_X^2 + \sigma_Y^2 \quad \#$$

Question 129:

From lecture notes, for 2-dim joint Gaussian R.V. (X,Y)

$$f_{XY}(x,y)=\frac{\exp\left\{-\dfrac{\left(\frac{x-m_X}{\sigma_X}\right)^2-2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right)+\left(\frac{y-m_Y}{\sigma_Y}\right)^2}{2(1-\rho^2)}\right\}}{2\pi\sigma_X\sigma_Y\left(\sqrt{1-\rho^2}\right)}$$

$$\Rightarrow \left.\begin{aligned}
&① \quad \sigma_X\sigma_Y\sqrt{1-\rho^2}=c\\
&② \quad -\frac{1}{2(1-\rho^2)\sigma_X^2}=\frac{-1}{2}\\
&③ \quad -\frac{1}{2(1-\rho^2)\sigma_Y^2}=\frac{-1}{2}\cdot 4=-2\\
&④ \quad \frac{\rho}{(1-\rho^2)\sigma_X\sigma_Y}=\frac{-1}{2}\cdot(-3)=\frac{3}{2}
\end{aligned}\right\}
\begin{aligned}
&\frac{④^2}{②③}: 4\rho^2=\frac{9}{4}\Rightarrow\rho=\frac{3}{4}\\
&\text{substitute } \rho \text{ into } ② \Rightarrow \sigma_X^2=\frac{16}{7}\\
&\qquad\qquad \rho \text{ into } ③ \Rightarrow \sigma_Y^2=\frac{4}{7}\\
&\text{From } ① \Rightarrow c=\frac{2}{\sqrt{7}}
\end{aligned}$$

Next, find the coefficients in front of x and y:

$$\left\{\begin{aligned}
&\frac{-1}{2(1-\rho^2)}\left(\frac{-2m_X}{\sigma_X^2}+\frac{2\rho m_Y}{\sigma_X\sigma_Y}\right)=\frac{-1}{2}\cdot(-2)\\
&\underbrace{\frac{-1}{2(1-\rho^2)}}_{=-\frac{8}{7}}\left(\frac{2\rho m_X}{\sigma_X\sigma_Y}+\frac{-2m_Y}{\sigma_Y^2}\right)=\frac{-1}{2}\cdot 3
\end{aligned}\right.
\Rightarrow
\left\{\begin{aligned}
&⑤\ -\frac{7}{8}m_X+\frac{21}{16}m_Y=-\frac{7}{8}\\
&⑥\ \frac{21}{16}m_X-\frac{7}{2}m_Y=\frac{-3}{2}\cdot\frac{-7}{8}=\frac{21}{16}
\end{aligned}\right.$$

From ⑤: $m_X=1+\frac{3}{2}m_Y$ substituted into ⑥: $m_Y=0 \Rightarrow m_X=1$

Thus, $E[X]=m_X=1$, $E[Y]=m_Y=0$

$Var(X)=\sigma_X^2=\frac{16}{7}$, $Var(Y)=\sigma_Y^2=\frac{4}{7}$

$Cov(X,Y)=\rho\sigma_X\sigma_Y=\frac{3}{4}\cdot\frac{4}{\sqrt{7}}\cdot\frac{2}{\sqrt{7}}=\frac{6}{7}$ #

Question 130:

$E[Y] = 0 \qquad \sigma_1 = 1 \qquad \sigma_2 = 2 \qquad E[X|Y] = \frac{1}{4}Y + 1$

$$E[X|Y] = m_1 + \rho_{X,Y}\frac{\sigma_1}{\sigma_2}(Y - m_2)$$

$$= m_1 + \rho_{X,Y}\left(\tfrac{1}{2}\right)(Y-0) = m_1 + \tfrac{1}{2}\rho_{X,Y}Y = \tfrac{1}{4}Y + 1$$

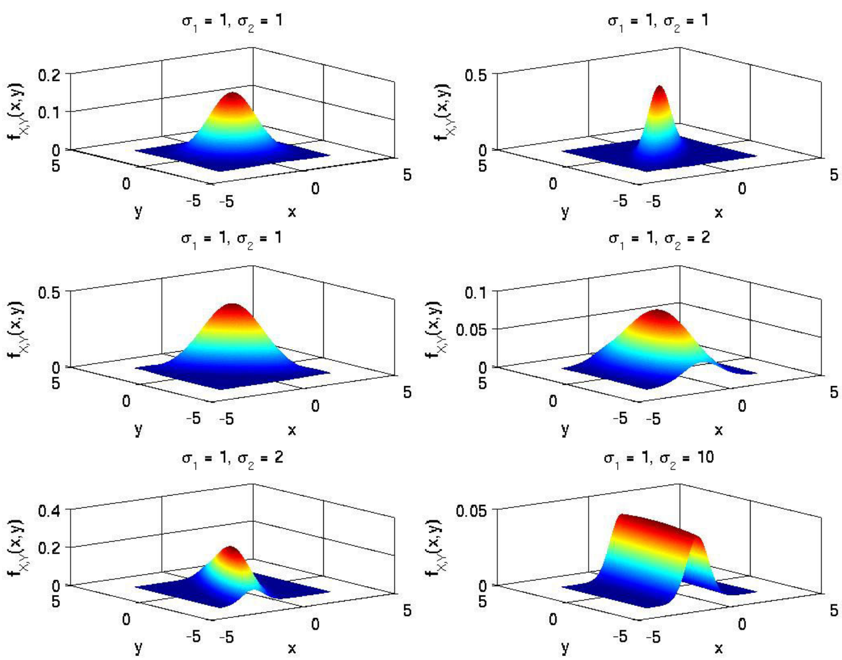
$$\Rightarrow m_1 = 1 \quad \& \quad \rho_{X,Y} = \tfrac{1}{2}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{X,Y}^2}}\, e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 + \left(\frac{y-m_2}{\sigma_2}\right)^2 - 2\rho_{X,Y}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right)\right]}$$

$$= \frac{1}{2\pi(1)(2)\sqrt{1-\frac{1}{4}}}\, e^{-\frac{1}{2(1-\frac{1}{4})}\left[\left(\frac{x-1}{1}\right)^2 + \left(\frac{y-0}{2}\right)^2 - 2\left(\frac{1}{2}\right)\left(\frac{x-1}{1}\right)\left(\frac{y-0}{2}\right)\right]}$$

$$= \frac{1}{2\pi\sqrt{3}}\, e^{-\frac{2}{3}\left[(x-1)^2 + \frac{y^2}{4} - (x-1)\left(\frac{y}{2}\right)\right]}$$

Question 131:



Question 132:

a: $E[Z] = E[X+Y] = m_X + m_Y = 1 - 2 = -1$

$E[Z^2] = E[X^2] + 2E[XY] + E[Y^2]$

$= (\sigma_X^2 + m_X^2) + 2(Cov(X,Y) + E[X]E[Y]) + (\sigma_Y^2 + m_Y^2)$

$= (2+1) + 2(1 + (1)(-2)) + (3+4)$

$= 3 + 2(-1) + 7 = 10 - 2 = 8$

$Var(Z) = E[Z^2] - (E[Z])^2 = 8 - (-1)^2 = 7$

b: $E[XZ] = E[X^2 + XY] = E[X^2] + E[XY] = (\sigma_X^2 + m_X^2) + (Cov(X,Y) + m_X m_Y)$

$= (2+1) + (1-2) = 3 - 1 = 2$

$Cov(X,Z) = E[XZ] - E[X]E[Z] = 2 - (1)(-1) = 3$

$\rho_{XZ} = \frac{Cov(X,Z)}{\sigma_X \sigma_Z} = \frac{3}{(\sqrt{2})(\sqrt{7})} = \frac{3}{\sqrt{14}}$

Question 133:

$P(X=-1) = P(X=1) = \frac{1}{2}$ $\quad N \sim N(0,1)$ $\quad X, N$ independent $\quad Y = X + N$

a: $E[Y] = E[X] + E[N] = 0$

$E[Y^2] = E[X^2] + 2E[XN] + E[N^2]$

$= \left((1)(\frac{1}{2}) + (1)(\frac{1}{2})\right) + 2(0)(0) + (1 + 0^2) = 2$

$Var(Y) = 2$

b: $Cov(X,Y) = E[XY] - E[X]E[Y]$

$= E[X^2] + E[X]E[N] - 0 = (1) + 0 = 1$

$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{1}{(1)\sqrt{2}} = \frac{1}{\sqrt{2}}$

c:

$f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x)$

$= \frac{1}{2}\delta(x-1)\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} + \frac{1}{2}\delta(x+1)\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}$

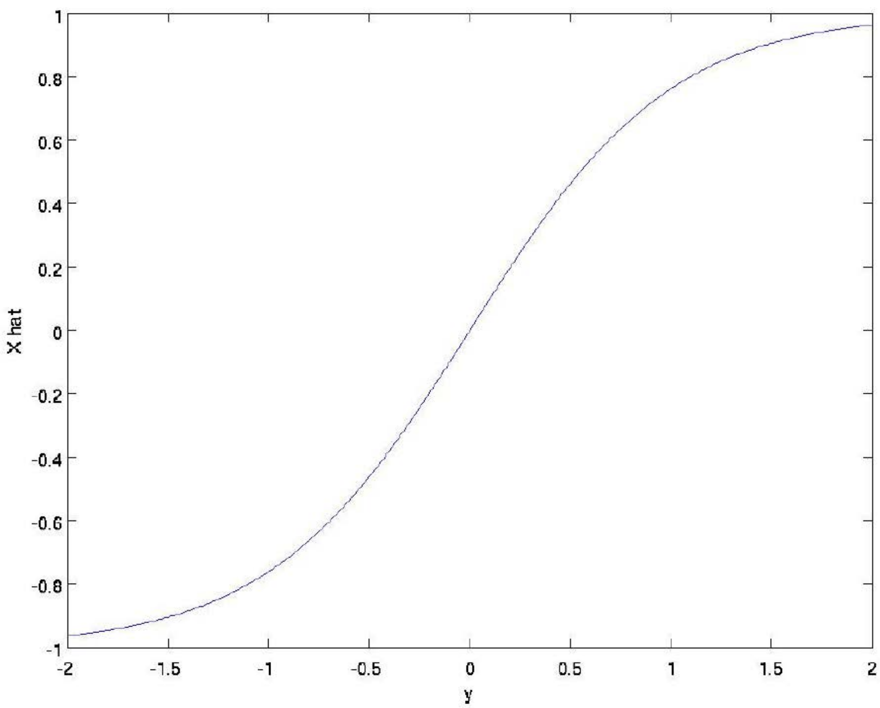
$\hat{X} = E[X|Y=y]$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}$

$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\delta(x-1)e^{-\frac{(y-1)^2}{2}} + \delta(x+1)e^{-\frac{(y+1)^2}{2}}}{e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y+1)^2}{2}}}$

$\hat{X} = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \frac{(1)e^{-\frac{(y-1)^2}{2}} + (-1)e^{-\frac{(y+1)^2}{2}}}{e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y+1)^2}{2}}}$

d:



e:

$$\hat{x} = \max_{x} f_{X|Y}(x|y) = \begin{cases} 1 & y > 0 \\ -1 & y < 0 \end{cases}$$

Question 134:

4 i) $E[X] = E[Y] = 0$

$E[X^2] = (1)(\frac{1}{3}) + (0)(\frac{1}{3}) + (1)(\frac{1}{3}) = \frac{2}{3} = E[Y^2]$

$Var(X) = Var(Y) = \frac{2}{3}$

$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\frac{2}{3}} = \frac{[(-1)(-1)(\frac{1}{6}) + (1)(-1)(\frac{1}{6})] - (0)(0)}{\frac{2}{3}} = \frac{3}{2}(0) = 0$

(ii) $E[X] = E[Y] = 0$

$Var(X) = Var(Y) = \frac{2}{3}$

$\rho_{XY} = 0$

(iii) $E[X] = E[Y] = 0$

$Var(X) = Var(Y) = \frac{2}{3}$

$\rho_{XY} = \frac{[(-1)(-1)(\frac{1}{3}) + (1)(1)(\frac{1}{3})] - (0)(0)}{\frac{2}{3}} = \frac{3}{2}(\frac{2}{3}) = 1$

a: $Y = \rho_{X,Y}\sigma_Y \frac{X - E[X]}{\sigma_X} + E[Y]$

(i) $\hat{Y} = \begin{cases} -\frac{1}{2} & x = \pm 1 \\ 0 & x = 0 \end{cases}$ (ii) $\hat{Y} = 0$ (iii) $\hat{Y} = X$

b: (i) $\hat{Y} = E[Y|x] = 0$

(ii) $\hat{Y} = E[Y|x] = 0$

(iii) $\hat{Y} = E[Y|x] = X$

c) $P_Y(j|X=k) = \frac{P[X=k|Y=j]P[Y=j]}{P[X=k]} = P_X(k|Y=j]$

Since $P[Y=j] = 1/3 = P[X=k]$

$\Rightarrow$ ML ad MAP estimates the same

(i)	(ii)	(iii)
$\hat{Y}(-1) = 0$ or -1	$\hat{Y}(-1) = 0, 1,$ or -1	$\hat{Y}(-1) = -1$
$\hat{Y}(0) = 1$	$\hat{Y}(0) =$ "	$\hat{Y}(0) = 0$
$\hat{Y}(+1) = 0$ or 1	$\hat{Y}(+1) =$ "	$\hat{Y}(+1) = +1$

Question 135:

$$f_{X,Y}(x,y) = \begin{cases} k(x+y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & o/w \end{cases}$$

$$\int_0^1\int_0^1 k(x+y)\,dx\,dy = k\int_0^1 \left(\frac{1}{2} + y\right) dy = k\left(\frac{1}{2}+\frac{1}{2}\right) \Rightarrow k=1$$

$$E[X] = E[Y] = \int_0^1\int_0^1 (x^2+xy)\,dx\,dy = \int_0^1\left(\frac{1}{3}+\frac{1}{2}y\right)dy = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

$$E[X^2] = E[Y^2] = \int_0^1\int_0^1 (x^3+x^2y)\,dx\,dy = \int_0^1\left(\frac{1}{4}+\frac{1}{3}y\right)dy = \frac{1}{4}+\frac{1}{6} = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{5}{12}$$

$$E[XY] = \int_0^1\int_0^1 (x^2y+xy^2)\,dx\,dy = \int_0^1\left(\frac{1}{3}y+\frac{1}{2}y^2\right)dy = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

$$Var(X) = Var(Y) = \frac{5}{12} - \frac{49}{144} = \frac{60-49}{144} = \frac{11}{144}$$

$$\rho_{XY} = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y} = \frac{\frac{1}{3}-\frac{49}{144}}{\frac{11}{144}} = \frac{144}{11}\left(\frac{48-49}{144}\right) = -\frac{1}{11}$$

a: $\hat{Y} = \rho_{XY}\sigma_Y \dfrac{X-E[X]}{\sigma_X} + E[Y] = -\dfrac{1}{11}\left(X-\dfrac{7}{12}\right)+\dfrac{7}{12}$

b: $\hat{Y} = E[Y|X] = \int_0^1 y\, f_{Y|X}(y|x)dy \qquad f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

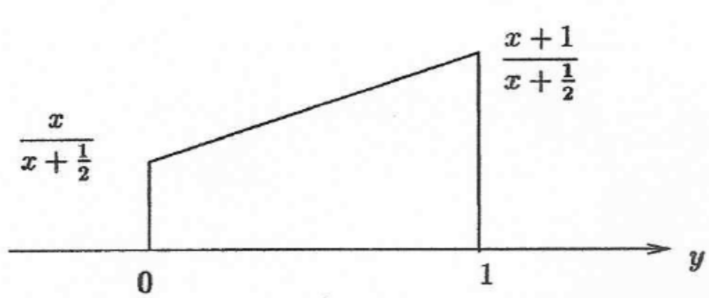
$$f_X(x) = \int_0^1 (x+y)\,dy = \frac{1}{2}+x$$

$$f_{Y|X}(y|x) = \frac{x+y}{x+\frac{1}{2}}$$

$$\hat{Y} = \int_0^1 y\frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\int_0^1 (y^2+xy)\,dy = \frac{1}{x+\frac{1}{2}}\left(\frac{1}{3}+x\right) = \frac{x+\frac{1}{3}}{x+\frac{1}{2}}$$

c

MAP



$$f(y|x) = \frac{x+y}{x+\frac{1}{2}} \qquad 0 \le y \le 1$$

$$\max\ y=1 \Rightarrow \hat{Y}=1$$

ML

$$f(x|y) = \frac{x+y}{y+0.5}, \quad \frac{\partial f(x|y)}{\partial x} = \frac{0.5-x}{(y+0.5)^2}$$

When x<0.5, the derivative is positive, so f(x|y) is increasing. Then maximum value is achieved when y=1. ML(x)=1.

When x>0.5, the derivative is negative, so f(x|y) is decreasing. Then maximum value is achieved when y=0. ML(x)=0.