ECE 302-003 Homework #12 Solution

Fall 2023

Question 118:

$\Theta \sim \text{Uniform}(0, 2\pi)$ $\quad X = \cos(\Theta) \quad Y = \sin(\Theta)$

a: $E[Y] = E[\sin(\Theta)] = \int_0^{2\pi} \sin(\theta) \frac{1}{2\pi} d\theta = \frac{1}{2\pi}(-\cos\theta)\Big|_0^{2\pi} = 0$

b: $E[XY] = E[\sin(\Theta)\cos(\Theta)] = \int_0^{2\pi} \frac{1}{2\pi} \sin\theta \cos\theta \, d\theta \qquad u = \sin\theta \, ' \, du = \cos\theta \, d\theta$

$= \frac{1}{2\pi} \frac{1}{2} \sin^2\theta \Big|_0^{2\pi} = 0$

c: $h(x) = E[Y \mid X = x]$

$$P_{Y|X=x}(y \mid X = x) = \begin{cases} \frac{1}{2} & y = \sin(\cos^{-1}(x)), \\ & y = -\sin(\cos^{-1}(x)) \\ 0 & \text{ow} \end{cases}$$

$h(x) = \frac{1}{2}\sin(\cos^{-1}(x)) - \frac{1}{2}\sin(\cos^{-1}(x)) = 0$

d: $E[h(X)] = E[Y] = 0$

Question 119:

a: $E[XY] = 0 \Rightarrow X$ & Y are orthogonal

b: $E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y = 0 - 0 = 0 \Rightarrow X$ & Y are uncorrelated

c: $\text{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = 0$

$\rho_{XY} = \frac{\text{cov}(X,Y)}{\sqrt{\sigma_X \sigma_Y}} = 0$

Question 120:

$$X \text{ \& } Y \text{ independent} \Rightarrow f_{XY}(x,y) = f_X(x) f_Y(y) \Rightarrow F_{XY}(x,y) = F_X(x) F_Y(y)$$

a: $P(a < X \leq b,\ Y > d) = (F_X(b) - F_X(a))(1 - F_Y(d))$

b: $P(a < X \leq b,\ c \leq Y < d) = (F_X(b) - F_X(a))(F_Y(d) - P(Y=d) - F_Y(c) + P(Y=c))$

c: $P(|X| < a,\ c \leq Y \leq d) = (F_X(a) - P(X=a) - F_X(-a))(F_Y(d) - F_Y(c) + P(Y=c))$

Question 121:

$X \sim N(0,1) \qquad Y \sim Uniform(0,3) \qquad X \text{ \& } Y$ independent

$$E[X^2 e^Y] = \int_{-\infty}^{\infty} \int_0^3 x^2 e^y \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{3}\, dy\, dx$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right]}_{E[X^2] = Var(X) + (E[X])^2 = 1} \left[\int_0^3 \frac{1}{3} e^y dy\right]$$

$$= \int_0^3 \frac{1}{3} e^y dy = \frac{1}{3} e^y \Big|_0^3 = \frac{1}{3}(e^3 - 1)$$

Question 122:

a: $f_X(x) = \begin{cases} \frac{1}{3} & x = -1, 0, 1 \\ 0 & ow \end{cases}$ $\quad f_Y(y) = \begin{cases} \frac{1}{3} & y = -1, 0, 1 \\ 0 & ow \end{cases}$

$f_{XY}(x,y) \neq f_X(x) f_Y(y) \Rightarrow X, Y$ not independent

b: $f_{XY}(x,y) = f_X(x) f_Y(y) \Rightarrow X, Y$ independent

c: $X = Y \Rightarrow X, Y$ not independent

Question 123:

a: $E[XY] = (-1)(-1)\frac{1}{6} + (-1)(0)\frac{1}{6} + (0)(1)\frac{1}{6} + (1)(-1)\frac{1}{6} + (1)(0)\frac{1}{6}$

$= \frac{1}{6} - \frac{1}{6} = 0$

$Cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[(X-0)(Y-0)] = E[XY] = 0$

uncorrelated, orthogonal, independent

b: $E[XY] = \frac{1}{9}[(-1)(-1) + (-1)(0) + (-1)(1) + (0)(-1) + (0)(0) + (0)(1) + (1)(-1) + (1)(0) + (1)(+1)]$

$= \frac{1}{9}[1 - 1 - 1 + 1] = 0$

$Cov(X,Y) = E[XY] = 0$

uncorrelated, orthogonal, independent

c: $E[XY] = \frac{1}{3}[(-1)(-1) + (0)(0) + (1)(1)] = \frac{2}{3}$

$Cov(X,Y) = E[X Y] = \frac{2}{3}$

$\rho_{X,Y} \neq 0$

correlated, not orthogonal, not independent

Question 124:

$$f_{XY}(x,y) = k(x+y) \quad \text{for} \quad 0 \le x \le 1,\ 0 \le y \le 1$$

$$\int_0^1 \int_0^1 k(x+y)\,dx\,dy = 1$$

$$k\int_0^1 \left[\tfrac{1}{2}x^2 + xy\right]_0^1 dy = k\int_0^1 \left(\tfrac{1}{2} + y\right) dy = k\left[\tfrac{1}{2}y + \tfrac{1}{2}y^2\right]_0^1 = k\left(\tfrac{1}{2} + \tfrac{1}{2}\right) = k$$

$$k = 1$$

$$E[XY] = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy = \int_0^1 \int_0^1 (x^2y + xy^2)\,dx\,dy$$

$$= \int_0^1 \left[\tfrac{1}{3}x^3y + \tfrac{1}{2}x^2y^2\right]_0^1 dy = \int_0^1 \left(\tfrac{1}{3}y + \tfrac{1}{2}y^2\right) dy$$

$$= \left[\tfrac{1}{6}y^2 + \tfrac{1}{6}y^3\right]_0^1 = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}$$

$$E[X] = \int_0^1 \int_0^1 (x^2 + xy)\,dx\,dy = \int_0^1 \left[\tfrac{1}{3} + \tfrac{1}{2}y\right] dy = \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{12}$$

$$E[Y] = \int_0^1 \int_0^1 (xy + y^2)\,dx\,dy = \int_0^1 \left[\tfrac{1}{2}y + \tfrac{1}{3}\right] dy = \tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12}$$

$$Cov(X,Y) = E\left[\left(X - \tfrac{7}{12}\right)\left(Y - \tfrac{7}{12}\right)\right] = E[XY] - \left(\tfrac{7}{12}\right)^2 = \tfrac{1}{3} - \tfrac{49}{144} = \frac{48-49}{144} = \frac{-1}{144}$$

not independent, not orthogonal, correlated

Question 125:

$Y = X + N$, where mean of X : $\mu_X = 0$
variance of X : σ_X^2
mean of N : $\mu_N = 0$
variance of N : σ_N^2

X and N are independent.

(a) $$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(X(X+N)) - E(X)E(X+N) \\ &= E(X^2) + \cancel{E(X)}\cancel{E(N)} - \cancel{E(X)}(E(X)+E(N)) \\ &= Var(X) + (\cancel{E(X)})^2 = \sigma_X^2 \quad \# \end{aligned}$$

$$\sigma_Y^2 = E(Y^2) - (\cancel{E(Y)})^2 = E(X^2) + 2E(X)E(N) + E(N^2)$$
$$= \sigma_X^2 + \sigma_N^2$$

Hence,
$$\rho_{XY} = \frac{\sigma_X^2}{\sigma_X\sqrt{\sigma_X^2+\sigma_N^2}} = \frac{\sigma_X}{\sqrt{\sigma_X^2+\sigma_N^2}} \quad \#$$

(b)

Let $f(a) = E[(X-aY)^2] = E[(X-a(X+N))^2]$

$$\begin{aligned} &= E[((1-a)X - aN)^2] = E[(1-a)^2X^2 - 2a(1-a)XN + a^2N^2] \\ &= (1-a)^2E[X^2] - 2a(1-a)E[X]E[N] + a^2E[N^2] \\ &= (1-a)^2\sigma_X^2 + a^2\sigma_N^2 \end{aligned}$$

$$\frac{d}{da} f(a) = -2(1-a)\sigma_X^2 + 2a\sigma_N^2 \overset{\text{set}}{=\!=} 0$$

$$\Rightarrow (1-a)\sigma_X^2 = a\sigma_N^2 \quad \Rightarrow a(\sigma_X^2 + \sigma_N^2) = \sigma_X^2$$

$$\Rightarrow a = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_N^2} \quad \#$$

(c)

$$f\left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}\right) = \left(\frac{\sigma_N^2}{\sigma_X^2+\sigma_N^2}\right)^2 \sigma_X^2 + \left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}\right)^2 \sigma_N^2$$

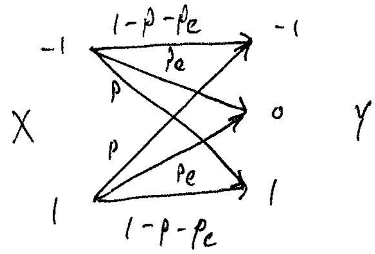
$$= \frac{\sigma_N^4\sigma_X^2 + \sigma_X^4\sigma_N^2}{(\sigma_X^2+\sigma_N^2)^2}$$

$$= \frac{\sigma_X^2\sigma_N^2(\sigma_N^2+\sigma_X^2)}{(\sigma_X^2+\sigma_N^2)^2}$$

$$= \frac{\sigma_X^2\sigma_N^2}{\sigma_X^2+\sigma_N^2} \quad \#$$

Question 126:

a:



$P(X=-1) = \frac{1}{4}$ $\quad$ $P(X=1) = \frac{3}{4}$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} = \frac{P_{Y|X}(y|x)\,P_X(x)}{P_Y(y)} = \begin{cases} \dfrac{\frac{1}{4}(1-p_e-p)}{\frac{1}{4}(1-p_e-p)+\frac{3}{4}p} & x=-1,\ y=-1 \\ \frac{1}{4} & x=-1,\ y=0 \\ \dfrac{\frac{1}{4}p}{\frac{1}{4}p+\frac{3}{4}(1-p-p_e)} & x=-1,\ y=1 \\ \dfrac{\frac{3}{4}p}{\frac{3}{4}p+\frac{1}{4}(1-p-p_e)} & x=1,\ y=-1 \\ \frac{3}{4} & x=1,\ y=0 \\ \dfrac{\frac{3}{4}(1-p-p_e)}{\frac{1}{4}+\frac{3}{4}(1-p-p_e)} & x=1,\ y=1 \end{cases}$$

Question 127:

$$f_{XY}(x,y) = (x+y) \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

a: $$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^1 f_{XY}(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \tfrac{1}{2}y^2\right]_0^1 = x + \tfrac{1}{2}$$

$$f_{Y|X}(y|x) = \frac{x+y}{x+\frac{1}{2}} \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

b $$P(Y>X|x) = P(Y>x) = \int_x^1 \frac{x+y}{x+\frac{1}{2}}\,dy = \left.\frac{xy+\frac{1}{2}y^2}{x+\frac{1}{2}}\right|_x^1 = \frac{1}{x+\frac{1}{2}}\left[\left(x+\tfrac{1}{2}\right) - \left(x^2+\tfrac{1}{2}x^2\right)\right]$$

$$= \begin{cases} 1 - \dfrac{\frac{3}{2}x^2}{x+\frac{1}{2}} & 0 \le x \le 1 \\ 0 & x < 0 \\ 1 & x > 1 \end{cases}$$

d: $$E[Y|X=x] = \int_0^1 y\,\frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\left[\tfrac{1}{2}y^2x + \tfrac{1}{3}y^3\right]_0^1 = \frac{\frac{1}{2}x+\frac{1}{3}}{x+\frac{1}{2}}$$

Extras:

$$E[Y] = \int_0^1\int_0^1 y\,(x+y)\,dx\,dy = \int_0^1 \left[\tfrac{1}{2}x^2y + xy^2\right]_0^1 dx$$

$$= \int_0^1 \left(\tfrac{1}{2}y + y^2\right)dy = \left[\tfrac{1}{4}y^2 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12}$$

$$E[E[Y|X]] = \int_0^1 h(x)\,f_X(x)\,dx = \int_0^1 \frac{\frac{1}{2}x+\frac{1}{3}}{x+\frac{1}{2}}\left(x+\tfrac{1}{2}\right)dx$$

$$= \int_0^1 \left(\tfrac{1}{2}x + \tfrac{1}{3}\right)dx = \left[\tfrac{1}{4}x^2 + \tfrac{1}{3}x\right]_0^1 = \tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12}$$