2-D Gaussian pdf

Note Title 4/4/2011

$$f_{XY}(x,y) = \frac{1}{2\pi\,\sigma_X\sigma_Y\left(\sqrt{1-\rho^2}\right)} \times e^{-\frac{\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2}{2(1-\rho^2)}}$$

HW12Q10 Prob 5.111

$$f_{XY}(x,y) = \frac{1}{2\pi \cdot c} e^{-\frac{1}{2}(x^2+4y^2-3xy+3y-2x+1)}$$

Find m_X, m_Y, σ_X^2, σ_Y^2, ρ, $Cov(X,Y)$

Ans: We first express it as

$$\frac{1}{2\pi \cdot c} e^{-\frac{1}{2}\left(1\times(x-a)^2+(-3)(x-a)(y-b)+4(y-b)^2\right)}$$

& find a, b, by inspection

x term: $-2a+3b=-2$

y term: $-8b+3a=3$ $\Rightarrow$ $b=0$ $a=1$

Constant term: $a^2-3ab+4b^2=1$ ✓

$$f_{XY}=\frac{1}{2\pi \times c}e^{-\frac{1}{2}((x-1)^2-3(x-1)y+4y^2)}$$

$$\Rightarrow \left.\begin{array}{l} ① \ \sigma_X\sigma_Y\sqrt{1-\rho^2}=c \\ ② \ \dfrac{1}{2(1-\rho^2)\sigma_X^2}=\dfrac{1}{2} \\ ③ \ \dfrac{1}{2(1-\rho^2)\sigma_Y^2}=\dfrac{4}{2} \\ ④ \ \dfrac{2\rho}{2(1-\rho^2)}\times\dfrac{1}{\sigma_X\sigma_Y}=\dfrac{3}{2} \end{array}\right\} \quad \begin{array}{l} \sigma_Y^2=4/7 \\ \sigma_X^2=\dfrac{16}{7} \\ c=\dfrac{2}{\sqrt{7}} \\ \rho=\dfrac{3}{4} \end{array}$$

$$④^2/②③ \Rightarrow \rho^2=\frac{9}{16} \xRightarrow{\text{by } ④} \rho=\frac{3}{4}$$

substitute ρ into ② $\Rightarrow$ $\sigma_X^2=\frac{16}{7}$

ρ into ③ $\sigma_Y^2=\frac{4}{7}$

from ① $\Rightarrow$ $c=\frac{2}{\sqrt{7}}$

$\Rightarrow m_X=1$, $m_Y=0$ $\sigma_X^2=\frac{16}{7}$, $\sigma_Y^2=\frac{4}{7}$ $c=\frac{2}{\sqrt{7}}$ $\rho=\frac{3}{4}$

$Cov(X,Y)=\frac{3}{4}\times\frac{8}{7}$