ECE 302-003 Homework #11 Solution

Fall 2023

Question 110:

$f_{Y|X}(y|x_0) = x_0 e^{-yx_0}$ for $y>0$ $\qquad f_X(x) = 1$ for $1<x<2$

a: $S_{X,Y} = (1,2) \times (0,\infty)$

b: $f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x)$

$$= \begin{cases} xe^{-xy} & 1<x<2,\ y>0 \\ 0 & \text{ow} \end{cases}$$

c: $P(X<1.5,\ Y\le 2) = \int_1^{1.5}\int_0^2 f_{XY}(x,y)\,dy\,dx$

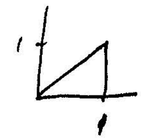
$$= \int_1^{1.5}\int_0^2 xe^{-xy}\,dy\,dx = \int_1^{1.5} -e^{-xy}\Big|_0^2 dx$$

$$= \int_1^{1.5}(1-e^{-2x})\,dx = \left[x+\tfrac{1}{2}e^{-2x}\right]_1^{1.5} = (1.5-1) + \tfrac{1}{2}(e^{-3}-e^{-2})$$

$$= \tfrac{1}{2} + \tfrac{1}{2}(e^{-3}-e^{-2})$$

Question 111:

a: $f_{XY}(x,y) = \begin{cases} k & 0 \le y \le x \le 1 \\ 0 & ow \end{cases}$

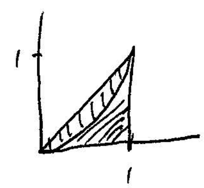


$= \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & ow \end{cases}$

b: $f_X(x) = \int_0^x 2\, dy = \begin{cases} 2x & 0 < x < 1 \\ 0 & ow \end{cases}$

$f_Y(y) = \int_y^1 2\, dx = \begin{cases} 2-2y & 0 < y < 1 \\ 0 & ow \end{cases}$

c: $p(Y < X^2) = \int_0^1 \int_0^{x^2} f_{XY}(x,y)\, dy\, dx$



$= \int_0^1 \int_0^{x^2} 2\, dy\, dx$

$= \int_0^1 2x^2\, dx$

$= \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$

Question 112:

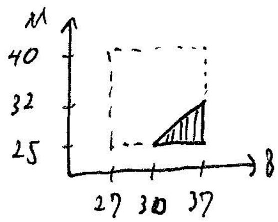
$B \sim \text{Uniform}(27, 37)$

$M \sim \text{Uniform}(25, 40)$

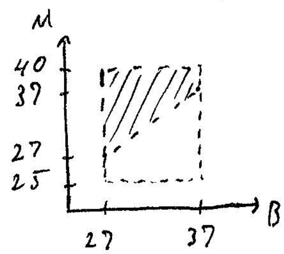
B & M independent

$$\frac{\text{Area of event}}{\text{total area}}$$

a: $P(M \leq B-5) = \frac{1}{2}(7)(7)\frac{1}{10} \cdot \frac{1}{15} = \frac{49}{300}$



b: $P(M \geq B) = \frac{1}{150}\left(\frac{1}{2}(10)(10) + (3)(10)\right) = \frac{50+30}{150} = \frac{8}{15}$



Question 113:

$X, Y \sim \text{Uniform}(-1, 1)$ iid

a: $P(X^2 < \frac{1}{2}, |Y| < \frac{1}{2}) = P(-\frac{1}{\sqrt{2}} < X < \frac{1}{\sqrt{2}}, -\frac{1}{2} < Y < \frac{1}{2})$

$= (\frac{1}{2})(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}})(\frac{1}{2})(\frac{1}{2} + \frac{1}{2}) = \frac{1}{2\sqrt{2}}$

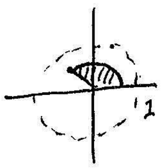
b: $P(4X < 1, Y < 0) = P(-1 < X < \frac{1}{4}, -1 < Y < 0)$

$= \frac{1}{4}(\frac{1}{4} + 1)(0 + 1) = \frac{5}{16}$

d: $P(\max(X, Y) < \frac{1}{3}) = P(X < \frac{1}{3}, Y < \frac{1}{3}) = P(-1 < X < \frac{1}{3}, -1 < Y < \frac{1}{3})$

$= \frac{1}{4}(\frac{1}{3} + 1)(\frac{1}{3} + 1) = \frac{16}{4(9)} = \frac{4}{9}$

Question 114:

a: 

$F_{R,\Theta}(r,\theta) = P(R \le r, \Theta \le \theta)$ for $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$= \left[\frac{\pi r^2}{\pi(1)^2}\right]\left[\frac{\theta}{2\pi}\right] = \frac{r^2\theta}{2\pi}$

$$F_{R,\Theta}(r,\theta) = \begin{cases} 0 & r \le 0 \text{ or } \theta < 0 \\ \frac{r^2\theta}{2\pi} & 0 < r \le 1 \text{ \& } \theta \in [0, 2\pi) \\ 1 & r > 1 \text{ and } \theta \ge 2\pi \end{cases}$$

$\frac{\theta}{2\pi}$ $r > 1$ and $\theta \in [0, 2\pi]$

r^2 $0 < r < 1$ and $\theta \ge 2\pi$

b: $F_R(r) = P(R \le r) = \begin{cases} 0 & r \le 0 \\ r^2 & 0 < r < 1 \\ 1 & r \ge 1 \end{cases}$

$$F_\Theta(\theta) = \begin{cases} 0 & \theta < 0 \\ \frac{\theta}{2\pi} & 0 \le \theta \le 2\pi \\ 1 & \theta > 2\pi \end{cases}$$

c: $P(R > \frac{1}{2}, 0 < \Theta \le \frac{\pi}{2})$

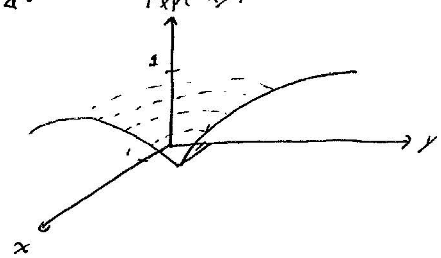
$= \left(1 - \left(\frac{1}{2}\right)^2\right)\left(\frac{\frac{\pi}{2}}{2\pi} - \frac{0}{2\pi}\right)$

$= \left(1 - \frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{3}{16}$

Question 115:

$$F_{X,Y}(x,y) = \begin{cases} (1-\frac{1}{x^2})(1-\frac{1}{y^2}) & x>1,\ y>1 \\ 0 & ow \end{cases}$$

a:



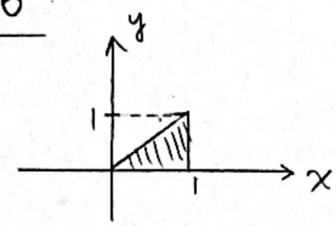
b: $F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} (1-\frac{1}{x^2}) & x>1 \\ 0 & x\le 1 \end{cases}$

$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} (1-\frac{1}{y^2}) & y<1 \\ 0 & y\ge 1 \end{cases}$

c: $P(X<3,\ Y\le 5) = F_{X,Y}(3,5) = (1-\frac{1}{9})(1-\frac{1}{25}) = (\frac{8}{9})(\frac{24}{25}) = \frac{64}{75}$

$P(X>4,\ Y>3) = (1-(1-\frac{1}{16}))(1-(1-\frac{1}{9})) = (\frac{1}{16})(\frac{1}{9}) = \frac{1}{144}$

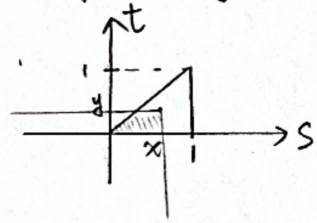
Question 116:



$$f_{XY}(x,y) = \begin{cases} 2 & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

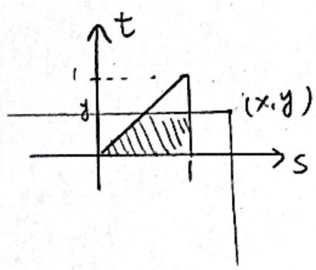
(a) 1° If $x \le 0$ or $y \le 0$, $F_{XY}(x,y) = 0$

2° If $0 \le y \le x \le 1$, $F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{ST}(s,t)\,ds\,dt$



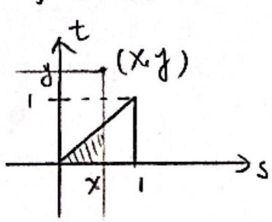
$$= \int_0^y \int_t^x 2\,ds\,dt$$
$$= \int_0^y 2(x-t)\,dt$$
$$= \left[2xt - t^2\right]_0^y$$
$$= 2xy - y^2 \quad \#$$

3° If $0 \le y \le 1$ and $x \ge 1$,



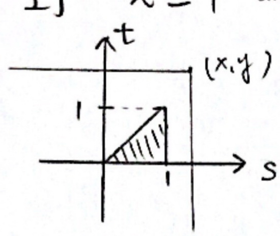
$$F_{XY}(x,y) = \frac{(1-y+1)\cdot y}{2} \cdot 2$$
$$= (2-y)y$$

4° If $0 \le x \le 1$ and $y > x$



$$F_{XY}(x,y) = \frac{x^2}{2} \cdot 2 = x^2$$

5° If $x \ge 1$ and $y > x$



$$F_{XY}(x,y) = 1$$

Hence,

$$F_{XY}(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \\ 2xy - y^2 & \text{if } 0 \le y \le x \le 1 \\ (2-y)y & \text{if } 0 \le y \le 1 \text{ and } x \ge 1 \\ x^2 & \text{if } 0 \le x \le 1 \text{ and } y > x \\ 1 & \text{otherwise.} \end{cases}$$

#

(b)

$$F_X(x) = F_{XY}(x, \infty) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x < 1 \\ 1 & , x \ge 1 \end{cases}$$

$$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 0 & , y < 0 \\ (2-y)y & , 0 \le y < 1 \\ 1 & , y \ge 1 \end{cases}$$ #

(c) $P(A) = P(X \le \frac{1}{2}, Y \le \frac{3}{4}) = (\frac{1}{2})^2 = \frac{1}{4}$ #

$$P(B) = P(\tfrac{1}{4} < X \le \tfrac{3}{4}, \tfrac{1}{4} < Y \le \tfrac{3}{4})$$

$$= F_{XY}(\tfrac{3}{4}, \tfrac{3}{4}) - F_{XY}(\tfrac{1}{4}, \tfrac{3}{4}) - F_{XY}(\tfrac{3}{4}, \tfrac{1}{4}) + F_{XY}(\tfrac{1}{4}, \tfrac{1}{4})$$

$$= (2 \cdot \tfrac{3}{4} \cdot \tfrac{3}{4} - (\tfrac{3}{4})^2) - (\tfrac{1}{4})^2 - (2 \cdot \tfrac{3}{4} \cdot \tfrac{1}{4} - (\tfrac{1}{4})^2) + (2 \cdot \tfrac{1}{4} \cdot \tfrac{1}{4} - (\tfrac{1}{4})^2)$$

$$= \tfrac{9}{16} - \tfrac{1}{16} - \tfrac{5}{16} + \tfrac{1}{16} = \tfrac{1}{4}$$ #

Question 117:

(a) for $j=-1$:

$P[X=j, Y\le y]$

$= P[X=-1, N-1\le y]$

$= P[X=-1, N\le y+1]$

$= (1-p)\cdot \frac{1}{0.5\sqrt{2\pi}} \int_{-\infty}^{y+1} e^{-x^2/2(0.25)}\, dx$

$= (1-p)\sqrt{\frac{2}{\pi}} \int_{-\infty}^{y+1} e^{-2x^2}\, dx$

for $j=1$:

$P[X=1, Y\le y]$

$= P[X=1, N+1\le y]$

$= P[X=1, N\le y-1]$

$= p\sqrt{\frac{2}{\pi}} \int_{-\infty}^{y-1} e^{-2x^2}\, dx$

(b) $p_X(-1) = 1-p \qquad p_X(1) = p$

$F_Y(y) = P[Y\le y \mid X=-1]P[X=-1] + P[Y\le y \mid X=1]P[X=1]$

$= P[N-1\le y](1-p) + P[N+1\le y](p)$

$= \int_{-\infty}^{y} \frac{(1-p)e^{-(y'+1)^2/2(0.25)}}{0.5\sqrt{2\pi}}\, dy' + \int_{-\infty}^{y} \frac{pe^{-(y'-1)^2/2(0.25)}}{0.5\sqrt{2\pi}}\, dy'$

$f_Y(y) = \frac{d}{dy}F_Y(y) =$

$= (1-p)e^{-2(y+1)^2}\cdot\sqrt{\frac{2}{\pi}} + pe^{-2(y-1)^2}\cdot\sqrt{\frac{2}{\pi}}$

(c) test for $X=1$:

$P[X=1 \mid Y>0]$

$$= \frac{P[X=1, Y>0]}{P[Y>0]}$$

$$= \frac{P[X=1, N+1>0]}{P[Y>0 \mid X=-1]P[X=-1] + P[Y>0 \mid X=1]P[X=1]}$$

$$= \frac{P[X=1, N>-1]}{(1-p)P[N>1] + pP[N>-1]}$$

$$= \frac{p\,Q(\frac{-1}{0.5})}{(1-p)\,Q(\frac{1}{0.5}) + p\,Q(\frac{-1}{0.5})}$$

$$= \frac{p\,Q(-2)}{(1-p)\,Q(2) + p\,Q(-2)}$$

$$= \frac{p\,(1-Q(2))}{(1-p)\,Q(2) + p\,(1-Q(2))}$$

from table:

$$= \frac{0.9772p}{0.0228 - 0.0228p + 0.9772p}$$

$$= \frac{0.9772p}{0.0228 + 0.9544p}$$

$P[X=1 \mid Y>0] > \frac{1}{2}$ when $0.0228 < p \leq 1$

and $< \frac{1}{2}$ when $0 \leq p \leq 0.0228$

$\therefore$ $X=1$ more likely when $p \in [0.0228, 1]$.