ECE 302-003 Homework #10 Solution

Fall 2023

Question 106:

$P(Y=k \mid X=x_0) = \frac{e^{-x_0}}{k!}(x_0)^k \quad k=0,1,2,\ldots \qquad X \sim \text{Geometric}(p)$

a: $S_{X,Y} = \{0,1,2,\ldots\} \times \{0,1,2,\ldots\}$

b: $P(Y=k, X=x_0) = P(Y=k \mid X=x_0)P(X=x_0)$

$$= \left[\frac{e^{-x_0}}{k!}(x_0)^k\right]\left[p(1-p)^{x_0}\right]$$

c: $P(X=x_0) = \sum_{k=0}^{\infty} P(Y=k, X=x_0)$

$$= p(1-p)^{x_0} \sum_{k=0}^{\infty} \frac{e^{-x_0}}{k!}(x_0)^k = p(1-p)^{x_0} \quad \text{for } x_0 = 0,1,2,\ldots$$

d: $P(X^2+Y^2 \leq 4) = P(X=0, Y=+2) + P(X=0, Y=+1) + P(X=0, Y=0) + P(X=1, Y=0)$

$+ P(X=1, Y=1) + P(X=2, Y=0)$

$$= 0+0+0+p(1-p)^1\frac{e^{-1}}{1}(1)^0 + p(1-p)^1\frac{e^{-1}}{1}(1)^1 + p(1-p)^2\frac{e^{-2}}{1}(2)^0$$

$$= 2p(1-p)e^{-1} + p(1-p)^2e^{-2}$$

Question 107:

$$f_{XY}(x,y) = e^{-\frac{x}{2}} y e^{-y^2} \qquad x>0,\ y>0$$

b: $$P(X^{\frac{1}{2}} > Y) = \int_0^\infty \int_0^{\sqrt{x}} f_{XY}(x,y)\,dy\,dx$$

$$= \int_0^\infty e^{-\frac{x}{2}} \int_0^{\sqrt{x}} y e^{-y^2}\,dy\,dx \qquad U = y^2$$

$$dU = 2y\,dy$$

$$= \int_0^\infty e^{-\frac{x}{2}} \int_0^x \frac{1}{2} e^{-u}\,dU\,dx$$

$$= \int_0^\infty e^{-\frac{x}{2}}\left(\frac{-1}{2}\right) e^{-u}\Big|_0^x dx = \int_0^\infty -\frac{1}{2} e^{-\frac{x}{2}}(e^{-x}-1)\,dx$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-\frac{x}{2}} - e^{-\frac{3}{2}x}\right)dx$$

$$= \frac{1}{2}\left[-2e^{-\frac{x}{2}} + \frac{2}{3}e^{-\frac{3}{2}x}\right]_0^\infty = \frac{1}{2}\left[2-\frac{3}{2}\right] = \frac{1}{2}\left(\frac{4-3}{2}\right) = \frac{1}{4}$$

c: $$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy$$

$$= e^{-\frac{x}{2}}\int_0^\infty y e^{-y^2}\,dy \qquad u = y^2 \quad dU = 2y\,dy$$

$$= \left[-\frac{1}{2}e^{-u}\Big|_0^\infty\right] e^{-\frac{x}{2}} = \frac{1}{2}e^{-\frac{x}{2}} \quad \text{for} \quad x>0$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx$$

$$= y e^{-y^2}\int_0^\infty e^{-\frac{x}{2}}\,dx = y e^{-y^2}\left[-2e^{-\frac{x}{2}}\right]_0^\infty$$

$$= 2y e^{-y^2} \quad \text{for} \quad y>0$$

Question 108:

$f_{XY}(x,y) = kx(1-x)y \quad \text{for } 0<x<1,\ 0<y<1$

a: $\int_0^1\int_0^1 f_{XY}(x,y)\,dx\,dy = 1$

$= \int_0^1\int_0^1 ky(x-x^2)\,dx\,dy = k\left[\int_0^1 y\,dy\right]\left[\int_0^1 (x-x^2)\,dx\right]$

$= k\left[\frac{1}{2}y^2\right]_0^1\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1 = k\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{k}{12} \quad \Rightarrow k=12$

c: $f_X(x) = \int_0^1 f_{XY}(x,y)\,dy = 12x(1-x)\int_0^1 y\,dy = 6x(1-x)\,y^2\Big|_0^1 = 6x(1-x) \quad \text{for } 0<x<1$

$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx = 12y\int_0^1 (x-x^2)\,dx = 12y\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1 = 2y \quad \text{for } 0<y<1$

d: $P(Y<X^{\frac{1}{2}}) = \int_0^1\int_0^{\sqrt{x}} f_{XY}(x,y)\,dy\,dx$

$= \int_0^1\int_0^{\sqrt{x}} 12(x-x^2)y\,dy\,dx = 12\int_0^1 (x-x^2)\,\frac{1}{2}y^2\Big|_0^{\sqrt{x}}\,dx$

$= 6\int_0^1 (x-x^2)(x-0)\,dx = 6\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1 = 6\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{6}{12} = \frac{1}{2}$

$$
\begin{aligned}
P[X<Y] &= \int_0^1\int_0^y f_{X,Y}(x,y)\,dxdy = \int_0^1\int_0^y 12\left(x-x^2\right)y\,dxdy \\
&= 12\int_0^1 \left[\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)y\right]_0^y dy = 12\int_0^1 \left(\frac{1}{2}y^2-\frac{1}{3}y^3\right)y\,dy \\
&= 12\int_0^1 \frac{1}{2}y^3-\frac{1}{3}y^4\,dy = 12\left[\frac{1}{8}y^4-\frac{1}{15}y^5\right]_0^1 \\
&= 12\left(\frac{1}{8}-\frac{1}{15}\right) = 12\cdot\frac{15-8}{120} \\
&= \frac{7}{10}
\end{aligned}
$$

Question 109:

(i) a: $k = \frac{1}{Area} = \frac{1}{\pi}$

b: $f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2}$ for $-1<x<1$

$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2}{\pi}\sqrt{1-y^2}$ for $-1<y<1$

c: $P(X>0,\ Y>0) = \frac{1}{4}$

(ii) a: $k = \frac{1}{4\,\frac{1}{2}(1)(1)} = \frac{1}{2}$

b: For $-1<x<0$,

$f_X(x) = \int_{-1-x}^{x+1} \frac{1}{2}\, dy = \frac{1}{2}(x+1) - \frac{1}{2}(-1-x) = \frac{1}{2}(2x+2) = x+1$

For $0<x<1$,

$f_X(x) = \int_{x-1}^{1-x} \frac{1}{2}\, dy = \frac{1}{2}[(1-x)-(x-1)] = \frac{1}{2}(2-2x) = 1-x$

$$f_X(x) = \begin{cases} x+1 & -1 \le x \le 0 \\ 1-x & 0 < x \le 1 \\ 0 & ow \end{cases}$$

For $-1<y<0$,

$f_Y(y) = \int_{-1-y}^{y+1} \frac{1}{2}\, dx = y+1$

For $0<y<1$,

$f_Y(y) = \int_{y-1}^{1-y} \frac{1}{2}\, dx = 1-y$

$$f_Y(y) = \begin{cases} y+1 & -1 \le y < 0 \\ 1-y & 0 \le y \le 1 \\ 0 & ow \end{cases}$$

c: $P(X>0,\ Y>0) = \frac{1}{4}$

(iii) a: $k = \frac{1}{\frac{1}{2}(1)(1)} = 2$

b: $f_X(x) = \int_0^{1-x} 2\, dy = 2-2x$ for $0<x<1$

$f_Y(y) = \int_0^{1-y} 2\, dx = 2-2y$ for $0<y<1$

c: $P(X>0,\ Y>0) = 1$