PURDUE

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1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
3. Add any additional pages used to the back of the exam before turning it in.
4. Ensure that all pages are facing the same direction.
5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam may contain some multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Please do not write on the back of each page. Any work on the back of the pages will not be scanned and thus will not be graded.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [12%, Work-out question] Consider two random variables (X, Y). The corresponding 2-dimensional CDF is described as follows.

$$F_{X,Y}(x,y) = \begin{cases} 1 - \frac{1}{x} - \frac{2}{y} + \frac{2}{xy} & \text{if } 2 \le x \text{ and } 3 \le y \\ 0 & \text{otherwise} \end{cases}$$

1. [2%] Find the expression of the marginal cdf $F_X(x)$.

2. [6%] Find the expression of the marginal pdf $f_X(x)$ and plot it for the range of $-1 < x < 5$.

   Hint: If you do not know how to solve this sub-question, you can instead plot the CDF of a binomial random variable W with parameter $n = 2$ and $p = 0.1$ for the range of $-2.5 < w < 5.2$. You will receive 4 points if your answer is correct.

3. [4%] Find the probability of $P(2 \le X, \text{ and } Y \le 4)$. Your answer should be something like $\frac{e^{-3}}{4} - \frac{e^{-8}}{\pi} + 0.2$

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This sheet is for Question 1.

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Question 2: [9%, Work-out question] Consider two random variables X and Y. We know that Y is uniform distribution within the interval $(1, 4.5)$ and given $Y = y_0$, X is an exponential random variable with $\lambda = \frac{1}{y_0+1}$.

1. [9%] Find the (1,1)-th moment of (X, Y).

   Hint 1: If you do not know what is the "(1,1)-th moment", you can solve the expectation $E(XY^2)$ instead. You will receive 7.5 points if your answer is correct.

   Hint 2: The table may be useful when solving this question.

Question 3: [12%, Work-out question] Consider two random variables X and Y with the joint pdf being

$$f_{XY}(x,y) = \begin{cases} \frac{1}{3}(x+y) & \text{if } 1 \le x < 2 \text{ and } 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

1. [12%] Compute the correlation coefficient ρ between X and Y.

   Hint 1: If you do not know the answer to this question, you can compute $E(X)$ and $\mathsf{Var}(X)$. You will receive 9 points if your answers are correct.

   Hint 2: You can leave your answers in the following form $\frac{(35/27)^4 - (21/55)^7 + \pi^5}{(25/7)^4 - (42/23)^7}$. There is no need to further simplify it.

   Hint 3: There is no "trick" in solving this question. You just have to be patient and carry out the computation. Please remember you don't need to write down the simplest answer. A form as in Hint 2 would be sufficient.

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This sheet is for Question 3.

Question 4: [12%, Work-out question] Consider a pair of joint Gaussian random variables (X, Y) with $m_X = 1$, $m_Y = 0$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and $\rho = \frac{3}{8}$.

1. [8%] We use (X, Y) to construct another random variable $W = 3X - 2Y$. Please write down the pdf $f_W(w)$ of the new random variable W.

   Hint 1: If you do not know how to solve Q4.1, please find the mean and variance of W, you will receive 6 points if your answer is correct.

2. [4%] Suppose there is a standard Gaussian random variable V that is independent of W. Find the probability $P(W + 3V > 5)$.

   Hint 2: The following values may be useful.

$$Q(0) = 0.5 \tag{2}$$
$$Q(0.4) = 0.3446 \tag{3}$$
$$Q(0.8) = 0.2119 \tag{4}$$
$$Q(1.2) = 0.1151 \tag{5}$$
$$Q(1.6) = 0.0548 \tag{6}$$
$$Q(2) = 0.0228 \tag{7}$$

   Hint 3: If you do not know how to solve Q4.2, please find the probability $P(X < -1)$. You will receive 3 points if your answer is correct.

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Question 5: [12%, Work-out question] Consider two continuous random variables (X, Y). We know that X is an exponential random variable with $\lambda_x = 1$ and Y is an exponential random variable with $\lambda_Y = 2$. We also know X and Y are independent.

1. [12%] Define $W = X + Y$, find the cdf $F_W(w)$ of W.

   Hint: If you do not know the answer to this question, you can find $P(\max(X, Y) < 3)$. You will receive 9 points if your answer is correct.

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This sheet is for Question 5.

Question 6: [12%, Work-out question] Consider two discrete random variables (X, Y). We know that X is binomial with $(n, p) = (3, 0.99)$. Given $X = x_0$, Y is binomial with parameters $(n, p) = (x_0, 0.6)$.

1. [3%] What does the acronym "MAP" detector stand for?

2. [4%] If we observe $Y = 1$, what is the value of the ML detector?

3. [5%] If we observe $Y = 1$, what is the value of the MAP detector?

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Question 7: [13%, Work-out question] Prof. Wang's car is in the shop recently, and he has to take a bus to school. He arrives at the bus stop at noon, 12:00pm, and the bus just left. We denote the arrival time of the next bus as X with the unit being minutes. For example, $X = 15$ means that the next bus arrives at 12:15:00 and $X = 3.25$ means that the next bus arrives at 12:03:15. Herein we use the format hh:mm:ss.

From historical data, Prof. Wang knows that X has the following pdf:

$$f_X(x) = \begin{cases} 0.5 \cdot 2e^{-2x} + 0.5 \cdot \frac{1}{10} & \text{if } 10 < x < 20 \\ 0.5 \cdot 2e^{-2x} & \text{if } 0 < x \le 10 \text{ or if } 20 \le x \end{cases} \tag{8}$$

Since Prof. Wang has missed the bus, he uses the time to solve the following probability questions.

1. [6%] What is the expected value of the "waiting time (unit: minutes)" before the next bus arrives?

   Hint: Using the formula table can save you some time so that you don't need to do integration by part.

2. [7%] Suppose Prof. Wang waits at the bus stop for 25 minutes and no bus has come yet. Find out the probability that Prof. Wang has to wait for at least *another* 5 minutes?

   Hint: Please first write down the "probability" in a mathematical form. For example, something like $P(X^2 < 5|X$ being a prime).

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Question 8: [18%, a yes/no question. There is no need to justify your answers]
　　Decide whether the following statements are true or false.

1. [2%] X and Y are uniformly distributed over a triangle with its three corners being $(-4, 0)$, $(2, 2)$ and $(2, -2)$. (For example, (X, Y) can be $(0.9, 0.77)$ or $(-1.9, -0.5)$, and so on so forth.) The random variables X and Y are orthogonal.

2. [2%] Consider three random variables X, Y, and Z. Suppose X and Y are positively correlated, and Y and Z are negatively correlated. Then X and Z must be negatively correlated.

3. [2%] X is a standard Gaussian random variable, Y is a standard Gaussian random variable. Then $X + Y$ is a Gaussian random variable with mean 0 and variance 2.

4. [2%] For any geometric random variable X with parameter $p = e^{-1}$ and any arbitrary discrete random variable Y. Regardless of whether X and Y are positively correlated, or negatively correlated, or uncorrelated, we always have $P(X + Y > 12) \geq P(Y > 12)$.

5. [2%] Suppose Θ is uniformly randomly distributed over the continuous interval $(0, \frac{\pi}{2})$. Construct two random variables $X = \cos(\Theta)$ and $Y = \sin(\Theta)$ from Θ. Then X and Y are negatively correlated.

6. [2%] It is possible to design the weight assignment of a random variable X that satisfies *simultaneously* (i) $E(X) = 10$ and (ii) $F_X(10) = 1$, where $F_X(x)$ is the corresponding cdf.

7. [2%] X is Bernoulli random variable with $p = 0.5$. Define $\mathsf{Var}(Y|x)$ as the conditional variance of Y given $X = x$. For example, $\mathsf{Var}(Y, 1)$ is the conditional variance of Y given $X = 1$. Also define $\mathsf{Var}(Y)$ as the variance of the (marginal) distribution of Y. Then we always have $\mathsf{Var}(Y) = 0.5(\mathsf{Var}(Y|0) + \mathsf{Var}(Y|1))$.

8. [2%] For any random variable X, we always have $P(e^{-0.1X} \geq \sqrt{2}) \leq \frac{E(e^{-0.1X})}{\sqrt{2}}$.

9. [2%] It is possible to construct a pair of joint Gaussian random variables (X, Y) satisfying $m_X = 40$, $m_Y = 1$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 9$ and $\mathsf{Cov}(X, Y) = 40$.

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This sheet is for Question 8.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1 - r^n)}{1 - r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1 - r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1 - r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6} \tag{7}$$

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\text{Var}(X) = p(1 - p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1 - p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\text{Var}(X) = np(1 - p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1 - p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\text{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.