* The most important function is the linear function

$$Z = aX + bY$$

$$W = cX + eY$$

We discuss the relationship between $m_Z$, $m_W$. & $Var(Z)$, $Var(W)$, $Cov(Z,W)$

① $m_Z = a m_X + b m_Y$

$m_W = c m_X + e m_Y$

② $Var(Z) = a^2 Var(X) + 2ab\, Cov(X,Y) + b^2 Var(Y)$

$\because E(Z^2) = E((aX+bY)^2) = E(a^2X^2 + 2abXY + b^2Y^2)$

This gives a hint of the above formula (esp. when all means = 0)

Similarly, $Var(W) = c^2 Var(X) + 2ce\, Cov(X,Y) + e^2 Var(Y)$

Q: $Cov(Z,W) = ?$

Ans: $Cov(Z,W) = ac\, Var(X) + (ae+bc)\, Cov(X,Y) + be\, Var(Y)$.

pf: Because

$$E(Z \cdot W) = E((aX+bY)(cX+eY))$$
$$= acE(X^2) + (ae+bc)E(XY) + beE(Y^2)$$

If $X$ and $Y$ are also indep.

$\Rightarrow Var(Z) = a^2 Var(X) + b^2 Var(Y)$

$\because Cov(X, Y) = 0$

HW12Q6

$X$ & $Y$ are indep & with means & variances $m_X, \sigma_X^2$, $m_Y, \sigma_Y^2$

Q: Find out the correlation between $X$ & $Y$

Ans: $E(XY) = Cov(X, Y) + m_X m_Y$

$= 0 + m_X m_Y$ # $\because$ indep.

Q: $Z = X + Y$. Find $m_Z$ & $E(Z^2)$

Ans: $E(Z) = E(X+Y) = E(X) + E(Y) = m_X + m_Y$

$E(Z^2) = E(X^2 + 2XY + Y^2)$

$= (\sigma_X^2 + m_X^2) + 2(m_X m_Y) + (\sigma_Y^2 + m_Y^2)$

$= \sigma_X^2 + \sigma_Y^2 + (m_X + m_Y)^2$

Q: $Var(Z)$

Ans: $= E(Z^2) - (E(Z))^2 = \sigma_X^2 + \sigma_Y^2$ #

* 2-dim Joint Gsn R.V. $(X, Y)$

$S_{XY}$ : {all real 2-dim vectors}.

five input parameters.

$m_X$, $m_Y$, $\sigma_X$, $\sigma_Y$, $\rho_{XY}$ (or just $\rho$)

$\rho_{XY} \rightarrow$ the correlation coeff

$$f_{XY}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \left(\sqrt{1-\rho^2}\right)} \times e^{-\frac{\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2}{2(1-\rho^2)}}$$

See p.229 for illustration

* Example : Prob 5.110

$$f_{XY}(x,y) = \frac{1}{2\pi \times c} e^{-2x^2 - y^2/2}.$$ is a joint Gsn.

Find $c$, $\sigma_X$, $\sigma_Y$, and $\rho_{X,Y}$, $Cov(X,Y)$

$\boxed{\text{Solved by inspection}}$

Ans: $\sigma_X \sigma_Y \sqrt{1-\rho^2} = c$

by inspecting the constant coeff

$2\rho\left(\frac{1}{\sigma_X}\right)\left(\frac{1}{\sigma_Y}\right) = 0$

by inspecting the $x \cdot y$ term

$\Rightarrow \boxed{\rho = 0}$

$\frac{1}{2(1-\rho^2)} \times \frac{1}{\sigma_X^2} = 2 \Rightarrow \boxed{\sigma_X^2 = \frac{1}{4}}$

$\sigma_X = \frac{1}{2}$

by inspection of the $x^2$ term

$\frac{1}{2(1-\rho^2)} \times \frac{1}{\sigma_Y^2} = \frac{1}{2} \Rightarrow \boxed{\sigma_Y^2 = 1}$ $\sigma_Y = 1$

by inspection of the $y^2$ term

$\boxed{c = \frac{1}{2} \times 1 \times \sqrt{1-0^2} = \frac{1}{2}}$

$\boxed{Cov(X,Y) = \rho \times \sigma_X \times \sigma_Y = 0}$

HW12Q10 Prob 5.111

$f_{XY}(x,y) = \frac{1}{2\pi \cdot c} e^{-\frac{1}{2}(x^2+4y^2-3xy+3y-2x+1)}$

Find $m_X$, $m_Y$, $\sigma_X^2$, $\sigma_Y^2$, $\rho$, $Cov(X,Y)$

Ans: We first express it as

$\frac{1}{2\pi c} e^{-\frac{1}{2}\left(1 \times (x-a)^2 + (-3)(x-a)(y-b) + 4(y-b)^2\right)}$

& find $a$, $b$, by inspection

$x$ term: $-2a+3b=-2$

$y$ term: $-8b+3a=3$ $\Rightarrow b=0 \quad a=1$

Constant term: $a^2-3ab+4b^2=1$ ✓

$$f_{XY}=\frac{1}{2\pi \times c}e^{-\frac{1}{2}((x-1)^2-3(x-1)y+4y^2)}$$

$$\Rightarrow \left.\begin{array}{l} ① \ \sigma_X\sigma_Y\sqrt{1-\rho^2}=c \\ ② \ \dfrac{1}{2(1-\rho^2)\sigma_X^2}=\dfrac{1}{2} \\ ③ \ \dfrac{1}{2(1-\rho^2)\sigma_Y^2}=\dfrac{4}{2} \\ ④ \ \dfrac{2\rho}{2(1-\rho^2)}\times\dfrac{1}{\sigma_X\sigma_Y}=\dfrac{3}{2} \end{array}\right\} \begin{array}{l} \sigma_Y^2=4/7 \\ \sigma_X^2=\dfrac{16}{7} \\ c=\dfrac{2}{\sqrt{7}} \\ \rho=\dfrac{3}{4} \end{array}$$

$$④^2/②③ \Rightarrow \rho^2=\frac{9}{16} \overset{\text{by } ④}{\Rightarrow} \rho=\frac{3}{4}$$

substitute $\rho$ into ② $\Rightarrow \sigma_X^2=\frac{16}{7}$

$\rho$ into ③ $\sigma_Y^2=\frac{4}{7}$

from ① $\Rightarrow c=\frac{2}{\sqrt{7}}$

$\Rightarrow m_X=1,\ m_Y=0\ \ \sigma_X^2=\frac{16}{7},\ \sigma_Y^2=\frac{4}{7}\ \ c=\frac{2}{\sqrt{7}}\ \ \rho=\frac{3}{4}$

$Cov(X,Y)=\frac{3}{4}\times\frac{8}{7}$

* 2-dim Joint Gsn R.V. $(X, Y)$

$S_{XY}$ : {all real 2-dim vectors}.

$$f_{XY}(x,y) = \frac{1}{2\pi\,\sigma_X\sigma_Y\left(\sqrt{1-\rho^2}\right)} \times e^{-\frac{\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2}{2(1-\rho^2)}}$$

* Properties of joint Gsn R.V.s.

① $E(X) = m_X$, $Var(X) = \sigma_X^2$

$Cov(X,Y) = \rho \cdot \sigma_X \cdot \sigma_Y$

② The marginal distribution of $X$ is Gsn, The marginal distribution of $Y$

is Gsn. Moreover, any linear combination of X & Y is (joint) Gsn

ex: $Z = 3X + 4Y$ is Gsn
$W = 2X - Y$ is Gsn
& (Z, W) are $\boxed{\text{joint Gsn}}$

Ex: X, Y are joint Gsn
with $m_X = 1$, $m_Y = 0$, $\sigma_X = 1$, $\sigma_Y = 2$
$\rho = -0.5$

Q: marginal pdf of $Y = ?$

Ans: by ② $\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi 2^2}} e^{-\frac{(y-0)^2}{2\times 2^2}}$

Q: $Z = 3X + 4Y$. Find $m_Z$, $\sigma_Z$.

Ans: $m_Z = 3m_X + 4m_Y = 3\times 1 + 4\times 0 = 3$

$$\sigma_Z^2 = 3^2 Var(X) + 2\times 3\times 4\, Cov(X,Y) + 4^2 Var(Y)$$
$$= 9\times 1 + 24\times(-0.5\times 1\times 2) + 16\times 4$$
$$= 49$$

is Gsn. Moreover, any linear combination of X & Y is (joint) Gsn

ex: $Z = 3X + 4Y$ is Gsn
$W = 2X - Y$ is Gsn
& $(Z, W)$ are $\boxed{\text{joint Gsn}}$

Ex: X, Y are joint Gsn
with $m_X = 1$, $m_Y = 0$, $\sigma_X = 1$, $\sigma_Y = 2$
$\rho = -0.5$

Q: marginal pdf of $Y = ?$

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Q: $Z = 3X + 4Y$. Find $m_Z$, $\sigma_Z$.

Ans: $m_Z = 3m_X + 4m_Y = 3 \times 1 + 4 \times 0 = 3$

$$\sigma_Z^2 = 3^2 \mathrm{Var}(X) + 2 \times 3 \times 4 \,\mathrm{Cov}(X, Y) + 4^2 \mathrm{Var}(Y)$$
$$= 9 \times 1 + 24 \times (-0.5 \times 1 \times 2) + 16 \times 4$$
$$= 49$$

* $Gsn \rightarrow \boxed{\text{Linear}} \xrightarrow{Gsn}$

* The benefit of working on a Gsn is that we only need to worry about the mean, variance, covariance of its input / output

We don't need to worry about $f_{XY}(x, y)$

Properties

③ If X and Y both are Gsn & X, Y are independent

$\Rightarrow$ (X, Y) are $\boxed{\text{joint Gsn}}$.

Exercise: Find a joint distribution (X, Y) s.t X & Y are both Gsn but (X, Y) is not joint Gsn.

④ Generally independent $\Longrightarrow$ uncorrelated

$\not\Longleftarrow$ Not vice versa.

but if X & Y are joint Gsn, then independent $\Longleftrightarrow$ uncorrelated

pf: Look at the Gsn joint pdf formula.

Ex: X & Y are standard Gsn. & X & Y are independent.

$Z = X + Y$

$W = X - Y$

Q: Are Z, W joint Gsn.

Ans: $\because$ (X, Y) are joint Gsn

$\therefore$ (Z, W) the linear combination of X and Y are joint Gsn.

Q: $m_Z,\ \sigma_Z,\ m_W,\ \sigma_W,\ Cov(Z,W) = ?$

Ans: $m_Z = m_X + m_Y = 0$

$m_W = m_X - m_Y = 0$

$$\sigma_Z^2 = \underbrace{\sigma_X^2}_{1} + \underbrace{2\,Cov(X,Y)}_{\because\ \rho=0\ \Rightarrow\ 0} + \underbrace{\sigma_Y^2}_{1}$$

by independence of X, Y

$= 2$,

$$\sigma_W^2 = \sigma_X^2 - 2\,Cov(X,Y) + \sigma_Y^2$$

$= 2$

$$Cov(Z,W) = E((Z-0)(W-0))$$

$= E(Z \cdot W)$

$= E((X+Y)(X-Y))$

$= E(X^2 - Y^2)$

$= E(X^2) - E(Y^2)$

$= (1+0^2) - (1+0^2)$

$= 0$

Q: Are Z, W independent?

Ans: Yes. $\because$ Cov(Z, W) = 0

Q: $f_{ZW}(z, w) = ?$

Ans: $$\frac{1}{2\pi\sqrt{2\times 2}} e^{-\frac{\left(\frac{z-0}{\sqrt{2}}\right)^2 - 0(\ )(\ ) + \left(\frac{w-0}{\sqrt{2}}\right)^2}{2\times(1-0^2)}}$$

$$= \frac{1}{2\pi\sqrt{2\times 2}} e^{-\frac{z^2+w^2}{2\times 2}}$$ #

Property ⑤ if X, Y are joint Gsn.

then $P(X \mid Y=y)$, the conditional distribution of X is also Gsn

with mean

$$m_X + \rho \times \frac{\sigma_X}{\sigma_Y} \times (y - m_Y)$$

variance $$\sigma_X^2(1-\rho^2)$$

see p. 281 for derivation

$X \to \boxed{\phantom{X}} \to Y$

## * Detection & estimation

The original quantity $X$ is unknown, we only observe $Y$. jointly $X$ & $Y$ are randomly distributed. Our goal is to derive the information of $X$ from the observation $Y$.

Ex: $X$: Signals at the base station
  $Y$: Signals received by the cellular phone.

Ex: $X$: Waveform in a concert.
  $Y$: Recorded MP3 signals.

Ex: $X$: The # of users login to a web server
  $Y$: The download speed of from my dormitory

Ex: $X$: The exact location of a missile
  $Y$: The radar output.

* Detection & Estimation $X \to \boxed{\quad\quad} \xrightarrow{Y}$

There are many different schems for detection & estimation with different performance complexity tradeoff.

Scheme 1: Maximum a posterior prob. (MAP) detector.

We first observe $Y = y_0$.
Find the $x$ with the largest condition prob.

prob $P(X = x \mid Y = y_0)$

Ex: similar to HWBQ8 Prob 6.68

| $X \backslash Y$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $\frac{1}{12}$ | $\frac{1}{6}$ | $0$ |
| $0$ | $0$ | $0$ | $\frac{1}{3}$ |
| $1$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $0$ |

Q: Find the MAP detector given $Y = y_0$.

Ans: Given $Y = 1$ $P(X = x \mid Y = 1)$

$$= \begin{cases} 0 & \text{if } x = -1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Given $Y=0$, $P(X=x \mid Y=0)$

$$= \begin{cases} \frac{1}{2} & \text{if } x=-1 \\ 0 & \text{if } x=0 \\ \frac{1}{2} & \text{if } x=1 \end{cases}$$

$Y=-1$ $\quad P(X=x \mid Y=-1)$

$$= \begin{cases} \frac{1}{4} & \text{if } x=-1 \\ 0 & \text{if } x=0 \\ \frac{3}{4} & \text{if } x=1 \end{cases}$$

$$\Rightarrow MAP(y) = \begin{cases} 1 & \text{if } y=-1 \\ \text{either } -1 \text{ or } 1 \text{ is fine} & \text{if } y=0 \\ 0 & \text{if } y=1 \end{cases}$$

☆ MAP has very strong performance (optimal)

as we always choose the most probable $X$ given the observation $Y=y$

☆ The drawback of MAP is its complexity

Both Finding conditional prob & finding the maximum are difficult to implement.

$\boxed{\text{Start from } P_X \to P_X \cdot P_{Y|X} \to P_{X|Y}}$

* Scheme 2: Maximum Likelihood (ML) detector.

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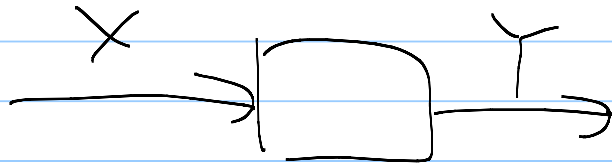
Recall that we have observed $Y = y_0$




and we are interested in infering $X$.

* Define the <u>likelihood</u> of $Y = y_0$

as $f_{\text{Likelihood}}(x) = P(Y = y_0 \mid X = x)$

The Maximum Likelihood (ML) detector thus outputs $x$ that has the largest

$$f_{\text{Likelihood}}(x) = P(Y = y_0 \mid X = x)$$

Comparison

Maximum A posteriori Prob (MAP) detector outputs $x$ that has the largest

$$P(X = x \mid Y = y_0)$$

Q: When do we use ML instead of MAP?

Ans: Sometimes we do not have the original marginal $P_X$. In this case, we just assume

$\boxed{P_X \text{ is uniform}}$

then $P(X=x \mid Y=y_0)$

$$= \frac{P(X=x, Y=y_0)}{P(Y=y_0)}$$

$$= \frac{P(Y=y_0 \mid X=x) \times \boxed{P(X=x)}}{\boxed{P(Y=y_0)}}$$

Do not depend on $x$

$$\text{maximizing}_{x} \; P(X=x \mid Y=y_0)$$

a posterior prob.

$$= \text{maximizing } P(Y=y_0 \mid X=x)$$

the likelihood

* ML can be viewed as a special case of MAP when $P_X$ (the prior) is uniform.

* Example: Conditional prob.

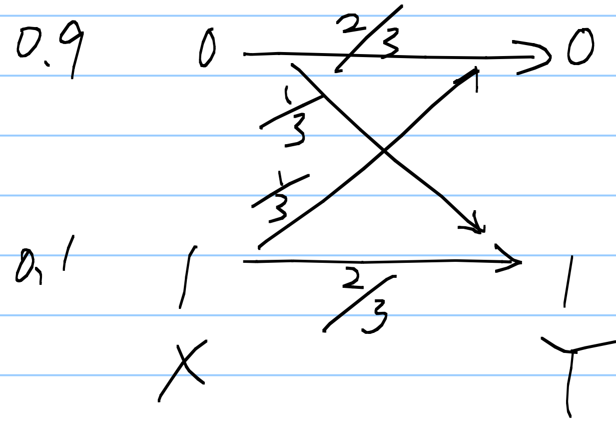
$P(Y=0 \mid X=0) = \frac{2}{3}$ $\quad$ $P(X=0) = 0.9$

$P(Y=1 \mid X=0) = \frac{1}{3}$ $\quad$ $P(X=1) = 0.1$

$P(Y=0 \mid X=1) = \frac{1}{3}$

$P(Y=1 \mid X=1) = \frac{2}{3}$

This is sometimes called the <u>binary symmetric channel</u>. 




Find MAP(y), ML(y)

Ans: ML(0): Comparing likelihood

$$P(Y=0 \mid X=0) = \frac{2}{3} \quad \checkmark$$

$$P(Y=0 \mid X=1) = \frac{1}{3}$$

ML(1) ... comparing

$$P(Y=1 \mid X=0) = \frac{1}{3}$$

$$P(Y=1 \mid X=1) = \frac{2}{3} \quad \checkmark$$

$$\Rightarrow ML(y) = \begin{cases} 0 & \text{if } y=0 \\ 1 & \text{if } y=1 \end{cases}$$

MAP(0): Comparing the conditional prob (posterior)

$$P(X=0 \mid Y=0) = \frac{0.9 \times \frac{2}{3}}{0.9\times\frac{2}{3} + 0.1\times\frac{1}{3}} = \frac{18}{19}$$

$$P(X=1 \mid Y=0) = \frac{0.1 \times \frac{1}{3}}{0.9\times\frac{2}{3} + 0.1\times\frac{1}{3}} = \frac{1}{19}$$

MAP(1) : comparing

$$P(X=0 \mid Y=1) = \frac{0.9\times\frac{1}{3}}{0.9\times\frac{1}{3}+0.1\times\frac{2}{3}} = \frac{9}{11}$$

$$P(X=1 \mid Y=1) = \frac{0.1\times\frac{2}{3}}{0.9\times\frac{1}{3}+0.1\times\frac{2}{3}} = \frac{2}{11}$$

$$\Rightarrow MAP(y) = \begin{cases} 0 & \text{if } y=0 \\ 0 & \text{if } y=1 \end{cases}$$

$\because P(X=0)=0.9$ is much more possible than $P(X=1)$.

MAP takes that into account while ML does not.

MAP detector has optimal performance, but higher complexity (finding $P_{X|Y} = \frac{P_{Y|X} P_X}{P_Y}$)

ML detector has slightly poorer performance & less complexity, (working on $P_{Y|X}$)