$$= \iint_{\square} 4xy\,dy\,dx + \iint 4xy\,dy\,dx$$

$$= \int_0^{0.5}\int_0^1 4xy\,dy\,dx + \int_{0.5}^{1}\int_0^{1-x} 4xy\,dy\,dx$$

$$=$$

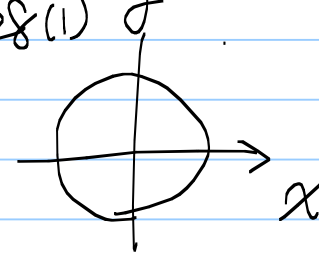
* From joint pdf to marginal pdf.

Knowing $f_{XY}(x,y)$ how to construct $f_X(x)$ (or $f_Y(y)$)

Ans: Integrating over uninterested variable.

Ex: HW11 Q3 Problem 5.28(i)

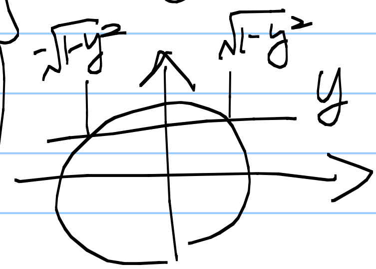
$$f_{XY}(x,y) = \begin{cases} \dfrac{1}{\pi} & \text{if} \\ 0 & \text{otherwise.} \end{cases}$$

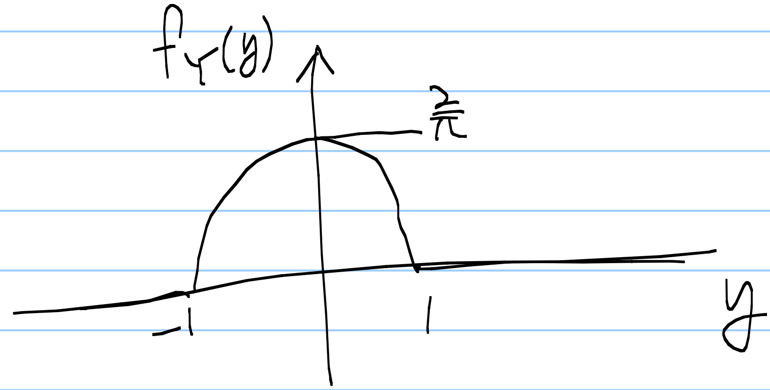


Find the marginal $f_Y(y)$.

Ans: $$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \begin{cases} 0 & \text{if } y < -1 \\ \displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx = \frac{2\sqrt{1-y^2}}{\pi} & \text{if } -1 \le y < 1 \\ 0 & \text{if } 1 \le y \end{cases}$$





Joint pdf $\longrightarrow$ marginal pdf ✓

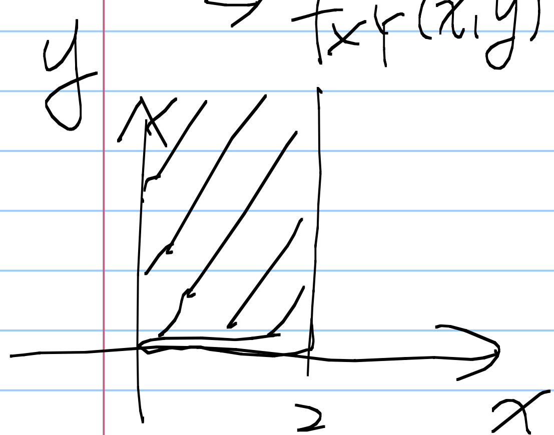
How about marginal + conditional pdf $\longrightarrow$ joint pdf ?

Ex: X is uniformly distributed on the interval $(0,2)$, and given $X=x$, Y is exponential with $\lambda = x$

Q: find the joint pdf of X, Y.

Ans: Joint pdf = marginal × conditional

$$\Rightarrow f_{XY}(x,y) = \begin{cases} \underbrace{\frac{1}{2}}_{\text{marginal}} \cdot \underbrace{x e^{-xy}}_{\text{conditional}} & \text{if } 0<x<2,\ 0<y \\ 0 & \text{otherwise} \end{cases}$$



* Independence: the marginals ≡ conditional

$$\Rightarrow f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

Ex: X is uniform on $(0,2)$

Y is exponential with $\lambda = 2$

X, Y are indep. Q: $f_{XY}(x,y) = ?$

Ans: $f_{XY}(x,y) = \begin{cases} \frac{1}{2}\, 2 \times e^{-2y} & \text{if } \begin{matrix} 0<x<2 \\ 0<y \end{matrix} \\ 0 & \text{otherwise} \end{cases}$

Summary

Note Title 3/21/2011

* Continuous 2-dim R.V.s.

$S_{XY} = \{$ all real vectors $\}$

W.A: Joint pdf $f_{XY}(x,y)$

the prob is the volume above the area of interest.

* Joint $\xrightarrow{\text{integration}}$ marginal
* Marginal · conditional $\longrightarrow$ joint

$$P(X=k)\cdot P(Y=h \mid X=k) = P(X=k, Y=h)$$

$$f_X(x) \cdot f_{Y|X}(y|x) = f_{XY}(x,y)$$

* Independence

$$\underbrace{f_{Y|X}(y|x)}_{\text{conditional}} = \underbrace{f_Y(y)}_{\text{marginal}}$$

$$\Rightarrow \quad f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

* Expectation for 2-dim conti R.V.

$E(g(X, Y))$ | say $E(X^2\sqrt{Y}+3Y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dy$$

HW11 Q10 Prob 5.58

X is standard Gsn

Y is uniform on [0, 3]

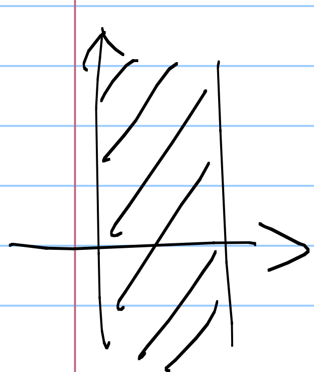
X & Y are indep

Q: Find $f_{XY}(x,y)$

Q: $E(X^2 e^Y) = ?$

Ans: $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{3} & \text{if } \begin{matrix} -\infty < x < x \\ 0 \le y \le 3 \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$



Q: $E(X^2 e^Y)$

$$= \int_{x=-\infty}^{\infty} \int_{y=0}^{3} x^2 e^y \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{3} \, dy \, dx$$

$$= \frac{1}{3} \int_{x=-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left[e^y \Big|_0^3 \right] dx$$

$$= \frac{e^3 - 1}{3} \int_{x=-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{e^3-1}{3} E(X^2) = \frac{e^3-1}{3} \left(Var(X) + E(X)^2 \right)$$

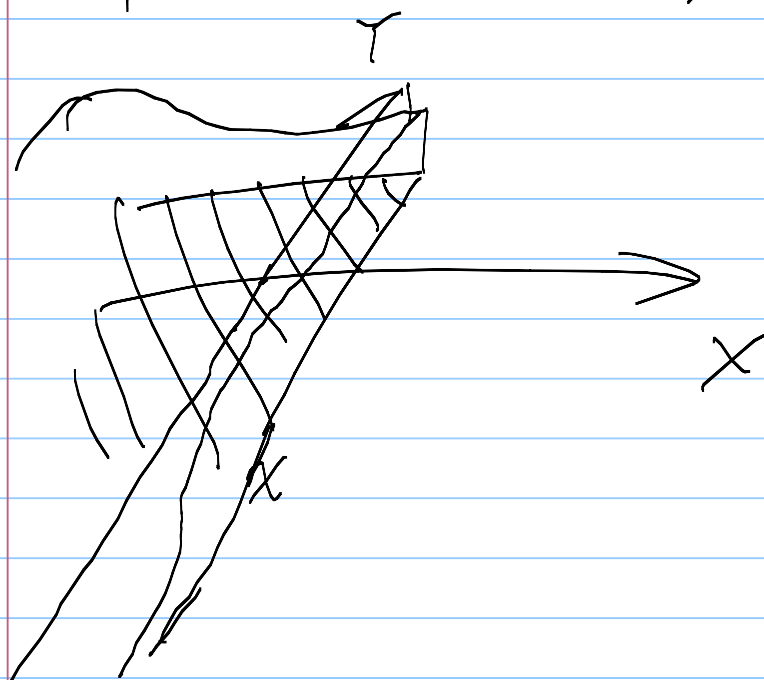
$$= \frac{e^3-1}{3} \times 1 \quad \#$$

$*$ Joint cdf: a unifying way to describe 2-dim discrete/conti R.Vs.

Joint cdf

$F_{XY}(x, y)$ is a function of two para. x & y.

$= P(X \le x,\ Y \le y)$



Ex: X and Y have a joint pmf

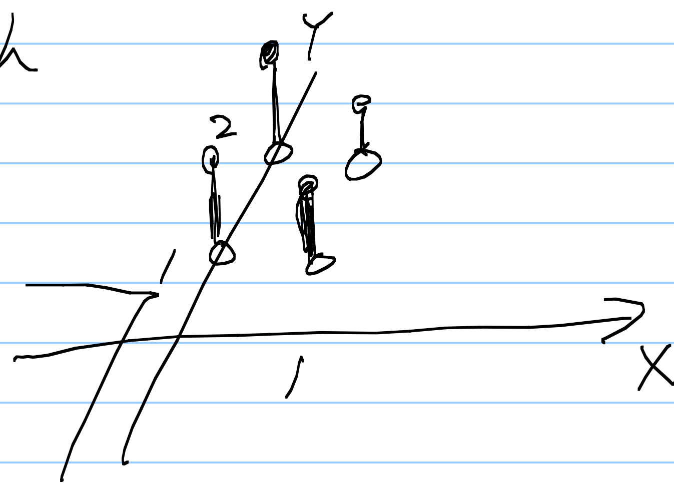
X \ Y	1	2
0	$\frac{1}{4}$	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{12}$

Find the joint cdf $F_{XY}(x, y)$

Ans: $F_{XY}(x, y) = P(X \le x,\ Y \le y)$

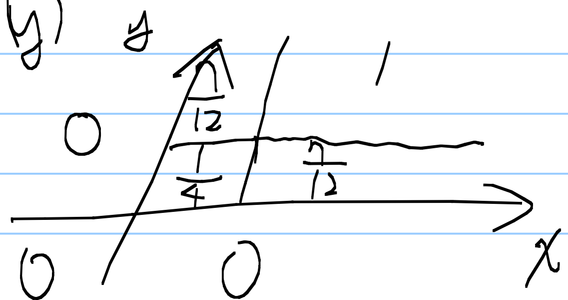
$$
= \begin{cases} 0 & \text{if } x<0 \\ 0 & \text{if } y<1 \\ p_{0,1} = \frac{1}{4} & \text{if } 0 \le x < 1, \ 1 \le y < 2 \end{cases}
$$

$p_{k,h}$



$$
\begin{cases} p_{0,1} + p_{1,1} = \frac{7}{12} & \text{if } 1 \le x, \ 1 \le y < 2 \\ p_{0,1} + p_{0,2} = \frac{7}{12} & \text{if } 0 \le x < 1, \ 2 \le y \\ p_{01} + p_{0,2} + p_{1,1} + p_{1,2} = 1 & \text{if } 1 \le x, \ 2 \le y \end{cases}
$$

$F_{XY}(x,y)$



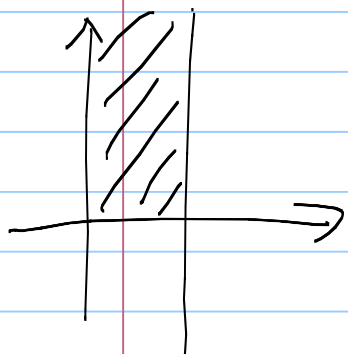
Note Title 3/21/2011

Ex: X is uniform on $(0,2)$

Y is exponential with $\lambda = 2$

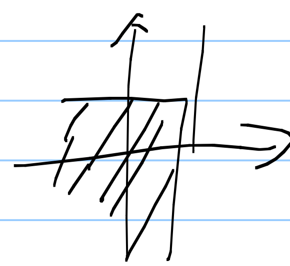
X, Y are indep. Q: Find $F_{XY}(x,y)$

Ans: $f_{XY}(x,y) = \begin{cases} \frac{1}{2} 2 \times e^{-2y} & \text{if } \begin{matrix} 0<x<2 \\ 0<y \end{matrix} \\ 0 & \text{otherwise} \end{cases}$



$F_{XY}(x,y) = P(X \le x, Y \le y)$

$= \begin{cases} 0 & \text{Case 1 if } x<0 \\ 0 & \text{Case 2 if } y<0 \\ & \text{Case 3 if } \begin{matrix} 0<x<2 \\ 0<y \end{matrix} \end{cases}$



$= \int_0^x \int_0^y \frac{1}{2} 2 \times e^{-2t} \, dt \, ds$

$= \int_0^x \frac{1}{2} (1 - e^{-2y}) \, ds$

$= \frac{x}{2} (1 - e^{-2y})$

Case 4

if $2 \le x$

$0 < y$

$$\int_0^2 \int_0^y \frac{1}{2} 2e^{-2t} \, dt \, ds$$

$$= (1 - e^{-2y}) \quad \#$$

* Properties of the joint cdf

① $F_{XY}(x, y) \ge 0$

② $F_{XY}(x, y) \le 1$

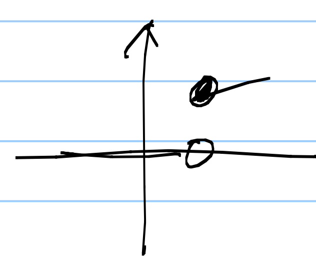
③ $F_{XY}(x, -\infty) = 0$, $\quad F_{XY}(\infty, \infty) = 1$.

$F_{XY}(-\infty, y) = 0$

④ $F_{XY}(x_1, y_1) \le F_{XY}(x_2, y_2)$

if $x_1 \le x_2$ and $y_1 \le y_2$

⑤ $F_{XY}(x, y)$ is continuous from the north & east.



$*\ F_{XY}(x,y) = P(X \le x, Y \le y)$

Note Title 3/25/2011

* Use $F_{XY}(x,y)$ to compute the prob.

The question will be of the form

Q: Given a
$$F_{XY}(x,y) = \begin{cases} 0 & \text{if } x<0 \text{ or } y<0 \\ \frac{x}{2}(1-e^{-2y}) & \text{if } 0\le x<2,\ 0\le y \\ (1-e^{-2y}) & \text{if } 2\le x,\ 0\le y \end{cases}$$

Q Find $P(X\le 3, Y\le 2)$

A: $= F_{XY}(3,2) = 1-e^{-4}$

Q: Find $P(X\le 3)$

A: $= F_{XY}(3,\infty) = 1-e^{-2\cdot\infty} = 1$

Q: Find $P(Y\le 5)$

A: $= F_{XY}(\infty,5) = 1-e^{-2\times 5} = 1-e^{-10}$

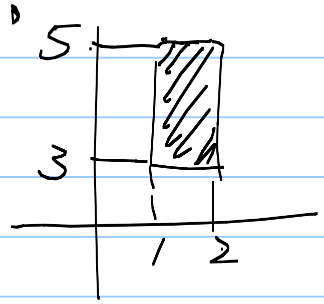
Q: Find $P(X<2, Y<5)$

A: $= F_{XY}(2^-,5^-) = \frac{2}{2}(1-e^{2\cdot 5}) = 1-e^{-10}$

it only matters only when either $x=2$ or $y=5$ is on the boundary of the cases of $F_{XY}(x,y)$

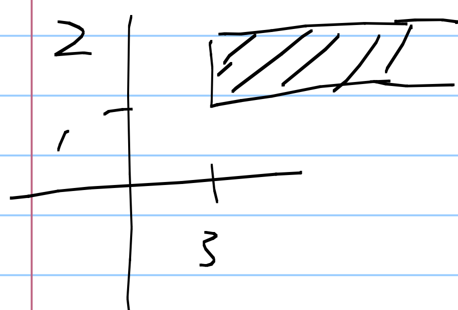
Q: $P(1 < X \le 2,\ 3 < Y \le 5) = ?$

A:



$$= F_X(2,5) - F_{XY}(2,3) - F_{XY}(1,5) + F_{XY}(1,3)$$

Q: $P(3 < X,\ 1 < Y \le 2)$



Ans: $F_{XY}(\infty, 2) - F_{XY}(3, 2) - F_{XY}(\infty, 1) + F_{XY}(3, 1)$

Q: $P(3 < X,\ 1 < Y < 2)$

A: $F_{XY}(\infty, 2^-) - F_{XY}(3, 2^-) - F_{XY}(\infty, 1) + F_{XY}(3, 1)$

* From joint cdf to marginal cdf

$$F_X(x) = P(X \le x) = F_{XY}(x, \infty)$$

$$F_Y(y) = P(Y \le y) = F_{XY}(\infty, y)$$

* From joint cdf $F_{XY}(x,y)$ to joint pdf $f_{XY}(x,y)$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,t)\,dt\,ds$$

$$f_{XY}(x,y) = \frac{d}{dx}\frac{d}{dy} F_{XY}(x,y)$$

* From joint cdf to marginal pdf

Route 1 : joint cdf $\xrightarrow{\text{diff}^2}$ joint pdf

$\xrightarrow{\text{integration}}$ marginal pdf

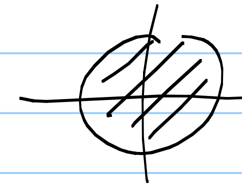
Route 2 : joint cdf $\longrightarrow$ marginal cdf

$\xrightarrow{\text{diff}}$ marginal pdf

HW11 Q11 Prob 5.18

(X, Y) is chosen uniformly randomly from a unit circle.

Q: $f_{XY}(x, y) = ?$

A: $f_{XY}(x,y) = \begin{cases} k & \text{if} \\ 0 & \text{otherwise.} \end{cases}$ 

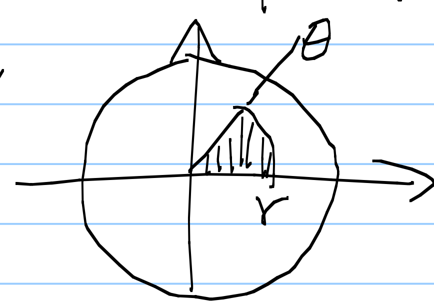
$\Rightarrow \quad k \times \pi = 1 \qquad k = \frac{1}{\pi}$

Q: Let $R = \sqrt{X^2 + Y^2}$
Θ = the angle

$\boxed{\text{Q: Find } f_{R\Theta}(r, \theta)}$

Find $F_{R,\Theta}(r, \theta)$ the cdf.

Ans: $F_{R,\Theta}(r, \theta) = P(R \leq r, \Theta \leq \theta)$

$= P\Big($ 

$\Big)$

$= \left(\frac{1}{2} r^2 \theta\right) \times k \qquad \Big\|$ Note: $k = \frac{1}{\pi}$

$$= \begin{cases} 0 & \text{Case 1 if } r<0 \text{ or } \theta<0 \\ \frac{1}{2} r^2 \theta \cdot \frac{1}{\pi} & \text{Case 2 if } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ r^2 & \text{Case 3 if } 0 \le r < 1,\ 2\pi \le \theta \\ \frac{\theta}{2\pi} & \text{Case 4 if } 1 \le r,\ 0 \le \theta < 2\pi \\ 1 & \text{Case 5 if } 1 \le r,\ 2\pi \le \theta \end{cases}$$

(b) $F_R(r) = ?$ the marginal cdf

$F_\Theta(\theta) = ?$ the marginal cdf

Ans: $F_R(r) = F_{R,\Theta}(r, \infty)$

$$= \begin{cases} 0 & \text{if } r<0 \quad (\text{Case 1}) \\ r^2 & \text{if } 0 \le r < 1 \quad (\text{Case 3}) \\ 1 & \text{if } 1 \le r \quad (\text{Case 5}) \end{cases}$$

$F_\Theta(\theta) = F_{R,\Theta}(\infty, \theta)$

$$= \begin{cases} 0 & \text{if } \theta<0 \quad \text{Case 1} \\ \frac{\theta}{2\pi} & \text{if } 0 \le \theta < 2\pi \quad \text{Case 4} \\ 1 & \text{if } 2\pi \le \theta \quad \text{Case 5} \end{cases}$$