$$= \frac{1}{j^2}\left(\frac{4j^2}{(2-j\omega)^3}\right)_{\omega=0}$$

$$= \frac{4}{8} = \frac{1}{2}$$

Q: $Var(X) = ?$

Ans: $E(X^2) - \left(E(X)\right)^2$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} = \frac{1}{\lambda^2}$$

(see Table 4.1)

* Characteristic function

$$\Phi_X(\omega) = E(e^{j\omega X})$$

Ex: X is Gaussian with μ, σ^2

Find $\Phi_X(\omega)$.

Ans: $\Phi_X(\omega) = E(e^{j\omega X})$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Again, it is the integration being the most difficult part.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 - 2\sigma^2 j\omega x)}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2(\mu + j\omega\sigma^2)x + (\mu + j\omega\sigma^2)^2 - (\mu + j\omega\sigma^2)^2 + \mu^2\right)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[(x - (\mu + j\omega\sigma^2))^2 - 2\mu j\omega\sigma^2 - (j\omega\sigma^2)^2\right]} dx$$

$$= \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x - (\mu + j\omega\sigma^2))^2}}_{=1} \times e^{+j\omega\mu - \frac{1}{2}\omega^2\sigma^2} dx$$

$$= e^{+j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

\#

Q $E(X) = ?$ $E(X^2) = ?$ $Var(X) = ?$

Ans:

① $\frac{d}{d\omega}\Phi_X(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}(j\mu - \omega\sigma^2)$

② $\frac{d^2}{d\omega^2}\Phi_X(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}\left((j\mu - \omega\sigma^2)^2 - \sigma^2\right)$

$\Rightarrow E(X) = \frac{1}{j}\left(① \Big|_{\omega=0}\right) = \frac{1}{j} \cdot j\mu = \mu$

$E(X^2) = \left(\frac{1}{j}\right)^2\left(② \Big|_{\omega=0}\right) = \mu^2 + \sigma^2$

$Var(X) = E(X^2) - \mu^2 = \sigma^2$

Note Title 2/25/2011

① Characteristic function

$$\Phi_X(\omega) = E(e^{j\omega X})$$

Moment theorem

$$E(X^n) = \left(\frac{1}{j^n}\right)\left[\frac{d^n}{d\omega^n}\Phi_X(\omega)\right]_{\omega=0}$$

$\Phi_X(\omega)$: Fourier transform of $f_X(x)$

② If we know that X is always non-negative $P(X<0)=0$, then we can get rid of the "j" by considering the following

$\boxed{\text{Moment generating function}}$

$X^*(s)$ is a function of a variable s

$$X^*(s) \triangleq E(e^{-sX})$$

Q: How to compute a moment generation function?

A: Simply evaluate $E(e^{-sX})$

Ex: Find the moment generation function of a Poisson R.V. w. para. α

Ans: $$\sum_{k=0}^{n} \frac{\alpha^k}{k!} e^{-\alpha} \cdot e^{-sk} = \sum_{k=0}^{\infty} \frac{(\alpha e^{-s})^k}{k!} e^{-(\alpha e^{-s})} \cdot e^{-\alpha} \cdot e^{(\alpha e^{-s})} = e^{-\alpha(1-e^{-s})}$$

* One-way of using $X^*(s)$ is

$$E(X^n) = (-1)^n \left[\frac{d^n}{ds^n} X^*(s)\right]_{s=0}$$

$\boxed{\text{the moment theorem}}$

---

③ If we know that X is not only non-negative $P(X<0)=0$, but also outputs only integers, $S=\{0,1,2,\dots\}$. The we have $\boxed{\text{probability generating function}}$

$G_X(z)$ is a function of variable z.

$$G_X(z) = E(z^X) = \sum_{k=0}^{\infty} z^k p_k$$

Ex: X is a fair dice

$G_X(z) = ?$

Ans: $$G_X(z) = \sum_{k=1}^{6} z^k \frac{1}{6}$$

$$= (z + z^2 + \cdots + z^6) \cdot \frac{1}{6}$$

Q: How to use $G_X(z)$? Two possibilities.

Ans: ① $$p_k = \frac{1}{k!}\left[\frac{d^k}{dz^k} G_X(z)\right]_{z=0}$$

(2) $E(\underbrace{X(X-1)(X-2)\cdots(X-(n-1))}_{\text{mulitiplcation of } N \text{ terms}})$

$$= \left[\frac{d^n}{dz^n} G_X(z)\right]\Big|_{z=1}$$

Ex: Suppose we know that for a Poisson R.V with para α

$$G_X(z) = e^{\alpha(z-1)}$$

Q: $E(X)$, $E(X(X-1))$, $E(X^2)$, $Var(X)$?

Ans: $E(X) = \frac{d}{dz} G_X(z)\Big|_{z=1} = \alpha e^{\alpha(z-1)}\Big|_{z=1}$

$= \alpha$

$E(X(X-1)) = \left[\frac{d^2}{dz^2} G_X(z)\right]\Big|_{z=1} = \alpha^2 e^{\alpha(z-1)}\Big|_{z=1}$

$= \alpha^2$

$E(X^2) = E(X(X-1)) + E(X) = \alpha + \alpha^2$

$Var(X) = E(X^2) - \alpha^2 = (\alpha + \alpha^2) - \alpha^2$

$= \alpha$

Functions of R.Vs.

Recall the advantage of considering R.V.s. is ① we can compute the weighted average

② we can easily generate new R.V from an old R.V.

Ex X is a R.V.

$Y_1 = X^2$ is another R.V.

$Y_2 = \frac{1}{X^2 - 1}$ is another R.V

Basically for any function $f(x)$

$Y = f(X)$ is a new R.V.

How to describe the new W.A of Y ?

Ans: Method 1: The most universal method is to ① Compute the cdf of Y first

$$F_Y(y) = P(Y \le y) = P(f(X) \le y)$$

② Use cdf to obtain pmf/pdf

12.1 Method 2: Sometimes we can compute the pmf / pdf of Y directly

* In the past, we have discussed different types of functions & the W.A of $Y = f(X)$. I.e. $X \longrightarrow \boxed{\text{System / function}} \xrightarrow{Y}$

Knowing the W.A of X, $\Rightarrow$ the W.A of Y

Ex: f is a "quantizer" HW6 Q2.

$f(0, x) = \max(x, 0)$ —— half-wave rectifier

Other important functions:

$f(x) = |x|$ —— full-wave rectifier

$f(x) = \min(x, 10)$ —— limiter/clipper

* You need some practice on computing the W.A

* The most important function is "linear functions" $\boxed{\text{the } ~~\text{of}~~ \text{ Y from X.}}$

$Y = aX + b$.

Ex: X is a geometric R.V with p.

$Y = 2X + 1$

Find the pmf of Y.

Ans: $p_k = P(Y = k)$

$= P(X = \frac{k-1}{2})$

$$= \begin{cases} 0 & \text{if } k \text{ is even} \\ 0 & \text{if } \frac{k-1}{2} < 0 \\ p(1-p)^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \\ & \text{\& } \frac{k-1}{2} \geq 0 \end{cases}$$

Basically the position of the prob mass has to be relocated.

for conti R.V. with $Y = aX + b$

We have a quick formula

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

if $a \neq 0$.

(see Example 4.31 for detailed derivation)

Q: If $a=0$, what is the pdf of Y?

Ans: $f_Y(y) = 1 \cdot \delta(y-b)$ ※.

Note Title 3/4/2011

* Expectation & Variance of $Y = aX + b$.

For any R.V. X (conti / discrete / mixed type)

We have $E(Y) = aE(X) + b$

$Var(Y) = |a|^2 Var(X)$

→ the center of the W.A

→ the expected "$\boxed{\text{squared}}$ distance" to the center

Ex: Continue from the example that X: geometric. $Y = 2X + 1$

Q: $E(Y)$, $Var(Y) = ?$ Ans: $E(Y) = 2E(X) + 1 = 2\frac{1}{p} + 1$, $Var(Y) = 4 \cdot \frac{1-p}{p^2}$

Q: Is Y a geometric R.V.?

Ans: No.

* Linear functions of Gaussian R.V.

Theorem: If X is a Gaussian R.V with μ_X, σ_X^2, and $Y = aX + b$.

then Y is also a Gaussian R.V with $\mu_Y = a\mu_X + b$

$\sigma_Y^2 = |a|^2 \sigma_X^2$

Therefore, whenever we see a Gsn R.V X with mean μ & variance σ^2, we should view it as a linear function of

$$X = \mu + \sigma Z$$

where Z is a Gsn with $\mu = 0$ (zero mean) & $\sigma^2 = 1$ (unit variance).

Such Z with zero-mean & unit variance is called the $\boxed{\text{standard}}$ Gsn R.V.

Ex: X is a Gaussian R.V with μ, σ^2. Find $P(X \leq 3)$ in terms of the prob of a standard Gsn R.V. Z

Ans: $P(X \leq 3)$

$$= P(\mu + \sigma Z \leq 3)$$

$$= P\left(Z \leq \frac{3 - \mu}{\sigma}\right)$$

Q: Why are we interested in a Standard Gsn?

Ans: We can construct a standard table for the cdf of Z. No need to construct multiple tables for different μ, σ^2 values.

In practice, we can compute a table of the cdf of Z.

$$\Phi_Z(z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi 1^2}} e^{-\frac{(x-0)^2}{2\cdot 1}} dx$$

$F_Z(z) \equiv \Phi_Z(z)$; Z is often omitted. (Don't be confused with the characteristic functions.)

Continue from the previous example,

Then $P(X \le 3)$ is obtained by looking up the value of $\Phi\left(\frac{3-\mu}{\sigma}\right)$ ※

Other "tables" for computing the prob of a Gssn R.V.

① Q functions: very popular in ECE.

$$Q(x) = 1 - \Phi(x)$$

$$= P(Z > x)$$

for the previous example

$$P(X \le 3) = 1 - Q\left(\frac{3-\mu}{\sigma}\right)$$ P.169

Other related "tables" $\boxed{\text{erf}}$ $\boxed{\text{erfc}}$ in statistics.

Ex: X is a Gsn with $\mu=2$ $\sigma^2=4$.

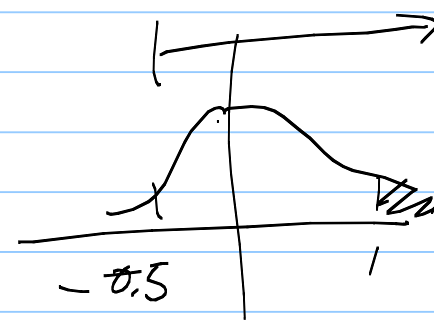
Q Find the prob $P(1 \le X \le 4)$ in terms of the Q function.

Ans: $P(1 \le X \le 4)$

$= P(1 \le 2 + 2Z \le 4)$

$= P(-0.5 \le Z \le 1)$

$= Q(-0.5) - Q(1)$



Q: You only know

$Q(0.5) = 0.309$ (Table 4.2)

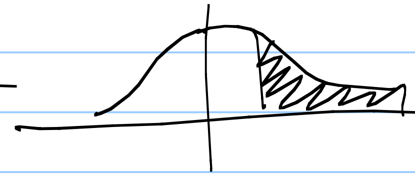
$Q(1) = 0.159$

Find the value of $P(1 \le X \le 4)$

Ans: the standard Gsn variable Z is symmetric

$\Rightarrow Q(-0.5) = 1 - Q(0.5)$

$= 1 -$



$\Rightarrow P(1 \le X \le 4) = (1 - Q(0.5)) - Q(1)$

$= 0.532$

In MATLAB, there is no Q function implemented. You can create the following .m file. fo

```
function y = q_ece(x)
y=0.5*erfc(x/sqrt(2));
```

Ex: X is a Gaussian with $\mu=1$, $\sigma^2=4$.

$Y=2X+1$. Find the prob

$$P(1\le Y\le 2 \text{ or } 3\le Y\le 4)$$

Ans: Y has mean $2\times 1+1=3$

variance $2^2\times 4=16=4^2$

$$\Rightarrow Y=3+4Z$$

$$P(1\le Y\le 2 \text{ or } 3\le Y\le 4)$$

$$=P(1\le 3+4Z\le 2,\ 3\le 3+4Z\le 4)$$

$$=P\left(\frac{-2}{4}\le Z\le \frac{-1}{4},\ 0\le Z\le \frac{1}{4}\right)$$

$$=Q\left(\frac{-2}{4}\right)-Q\left(\frac{-1}{4}\right)+Q(0)-Q\left(\frac{1}{4}\right)$$

$$=\left(1-Q\left(\frac{1}{2}\right)\right)-\left(1-Q\left(\frac{1}{4}\right)\right)+Q(0)-Q\left(\frac{1}{4}\right)$$

$$=Q(0)-Q\left(\frac{1}{2}\right)=\frac{1}{2}-Q\left(\frac{1}{2}\right)$$

\#

Note Title 3/2/2011

= Summary

* For any R.V. X. & $Y = aX + b$.

$\mu_Y = a\mu_X + b$, $Var(Y) = a^2 Var(X)$

* Gaussian R.V + linear transformation

Two important conclusions.

① If X is a Gsn with μ_X, σ_X^2 and $Y = aX + b$, then Y is also a Gsn with $\mu_Y = a\mu_X + b$

$$\sigma_Y^2 = a^2 \sigma_X^2$$

② If X is a Gsn with μ_X, σ_X^2 then X can be viewed as a generated by $X = \mu_X + \sigma_X Z$ where Z is a mean 0, variance 1 Gsn R.V, called the standard Gsn.

③ Computing the prob of Z is achieved by table look-up. $F_Z(z) \triangleq \Phi(z) = P(Z \le z)$

or $Q(x) = P(Z > x) = 1 - \Phi(z)$

So far, we assume that we know the W.A <u>completely</u> (say pdf/pmf/cdf/charact... moment generating/prob

But in many cases we only know part of the W.A. Can we still do some meaningful implication?

Probability Bounds:

① Union bounds.

If we know ① $P(0<X<3)$ and ② $P(1<X<5)$.

Q: what is the estimate of

$$P(0<X<3 \text{ or } 1<X<5)$$

Ans: $\max(P(0<X<3), P(1<X<5)) \leq ? \leq P(0<X<3) + P(1<X<5)$

i.e., max (P(A),P(B)) <= P(A or B) <= P(A)+P(B)

② The Markov Inequality

We know ① $P(X<0)=0$

② $E(X)=m$ say $m=5$

We want to estimate $P(X\geq a)$ say $a=3$.