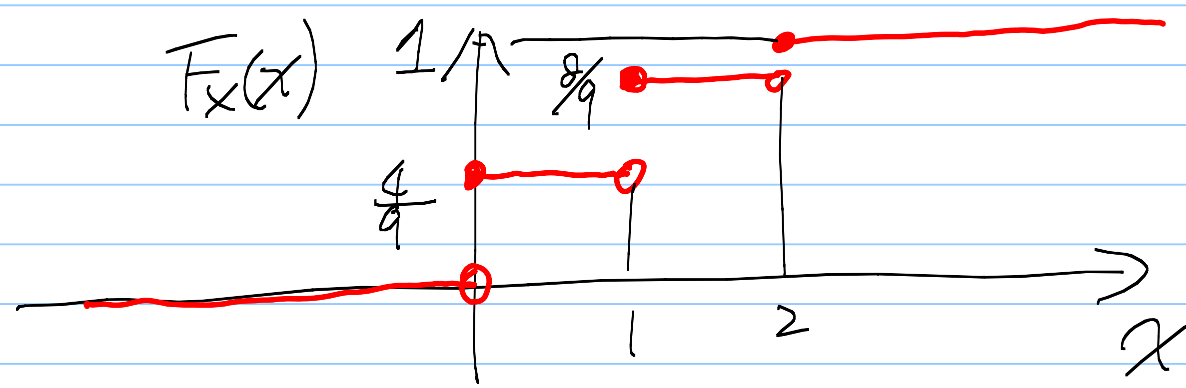
Ex: X is a binomial R.V with $n=2$, $p=\frac{1}{3}$

Find the cdf $F_X(x)$, Plot it.

Ans: $F_X(x) = P(X \le x)$

$$= \begin{cases} 0 & \text{if } x<0 \\ p_0 = \frac{4}{9} & 0 \le x < 1 \\ p_0+p_1 = \frac{8}{9} & 1 \le x < 2 \\ p_0+p_1+p_2 = 1 & 2 \le x \end{cases}$$



$\boxed{\text{II}}$ From cdf $F_X(x)$ back to p_k.

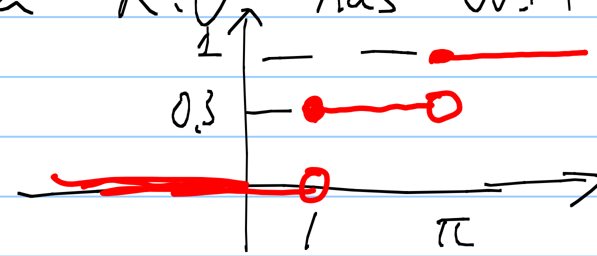
Ans: For each k value, p_k is the jump from $F_X(k-0.0001)$ to $F_X(k)$ (or sometimes we write it as $p_k = F_X(k) - F_X(k^-)$)

Ex: I can say a R.V. has W.A

① $S = \{1, \pi\}$

$P(X=1) = 0.3$

$P(X=\pi) = 0.7$



Or ② $F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.3 & \text{if } 1 \le x < \pi \\ 0.3+0.7 & \text{if } \pi \le x \end{cases}$

Both ① and ② describe the same W.A.

As you can see, the pmf ① indeed corresponds to the jump in the cdf ②

* Summary:

① Given the pmf P_k, we can find the CDF $F_X(x)$ by counting.

② Given the CDF $F_X(x)$, we can find the pmf by location & the magnitude of the jumps.

Conti R.Vs:

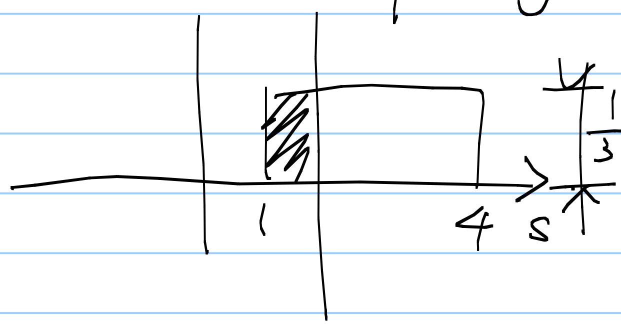
$\boxed{III}$ From pdf $f_X(x)$ to cdf $F_X(x)$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(s)\,ds$$

Ex: X is a uniform R.V over (1, 4)

What is the cdf of X.

Ans:
$$f_X(s) = \begin{cases} \frac{1}{4-1} & \text{if } 1 < s < 4 \\ 0 & \text{otherwise} \end{cases}$$



$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \int_1^x \frac{1}{3}\,ds = \frac{x-1}{3} & \text{if } 1 \le x < 4 \\ \int_1^4 \frac{1}{3}\,ds = 1 & \text{if } 4 \le x \end{cases}$$

$\boxed{\text{IV}}$ How to start from a cdf $F_X(x)$ to derive the pdf $f_X(x)$?

Ans: $\because$ $F_X(x) = \int_{-\infty}^{x} f_X(s)\,ds$

$\therefore f_X(x) = \frac{d}{dx} F_X(x)$ $\boxed{\text{Differentiation}}$

Exercise: Find & plot the cdf $F_X(x)$ for an exponential R.V.

Properties of a cdf $F_X(x)$

① $0 \le F_X(x) \le 1$.

② $\lim_{x \to \infty} F_X(x) = 1$

③ $\lim_{x \to -\infty} F_X(x) = 0$

④ $F_X(x)$ is non-decreasing

I.e. if $a < b$ then $F_X(a) \le F_X(b)$

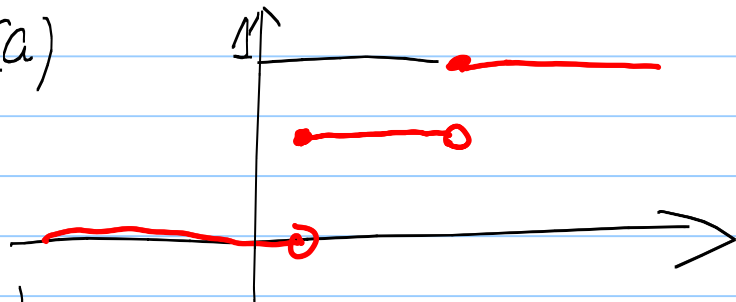
⑤ $F_X(x)$ is conti from the right.

Namely $F_X(x^-)$ may / may not be $F_X(x)$

but $F_X(x^+) = F_X(x)$

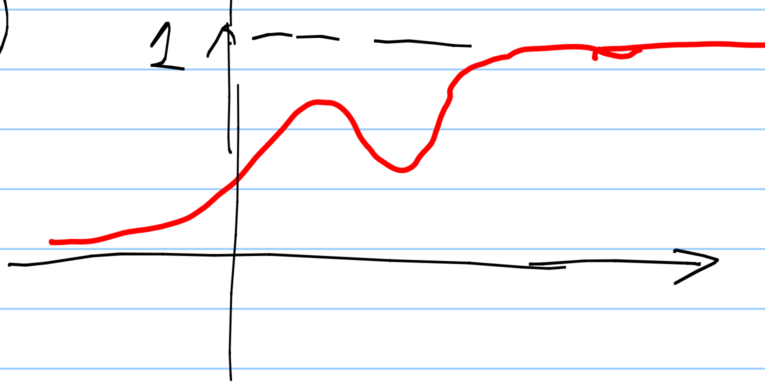
Ex: Which of the following figures can be a valid cdf $F_X(x)$

(a)

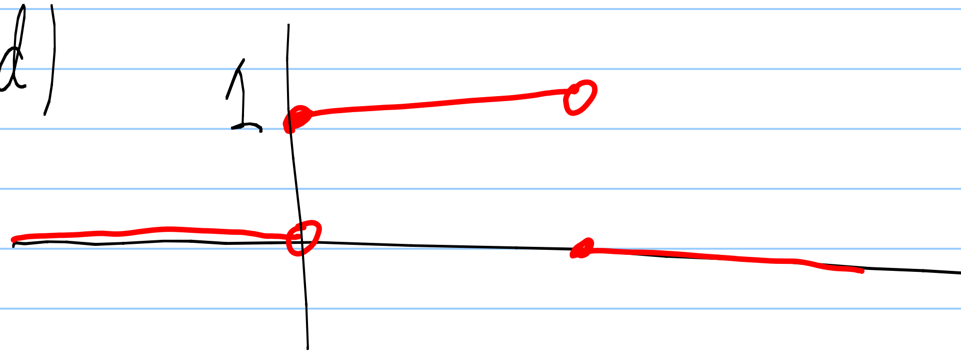


(b)

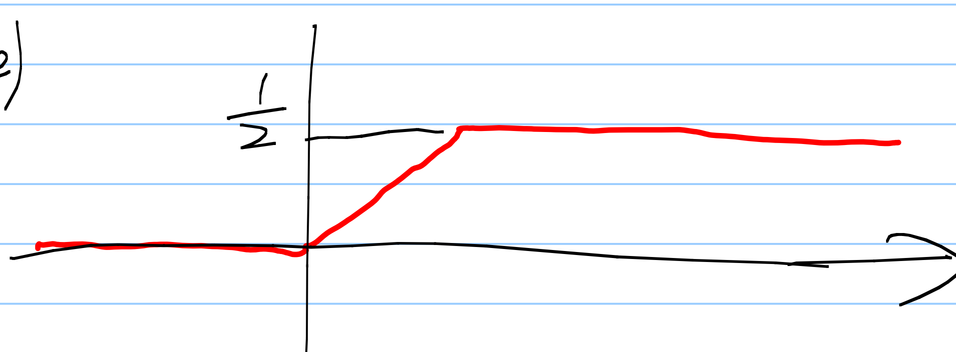
(c)



(d)



(e)



The $\boxed{\text{cumulative distribution function}}$ is less straightforward than the pmf & pdf ex: $F_X(x)$ the x can be fractional even when X is a discrete R.V.

However there are many advantages to use a cdf $F_X(x)$

* ① Applies to both discrete & conti R.V.

② Can be used to compute probabilities

Ex: Given a R.V. X with cdf $F_X(x)$

for example $F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$

(or $F_X(x) = U(x)$)

Find

Q: $P(X \le a) = ?$

Q: $P(a < X) = ?$

Q: $P(a < X \le b) = ?$

Q: $P(X < a) = ?$

Q: $P(a \le X) = ?$

Q: $P(a \le X \le b) = ?$

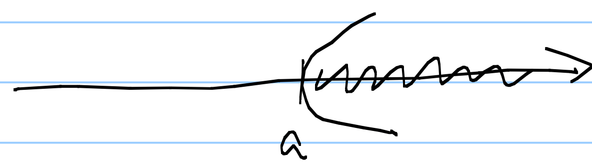
in terms of $F_X(x)$

Q: $P(X=a) = ?$ Q: $P(a<X<b) = ?$

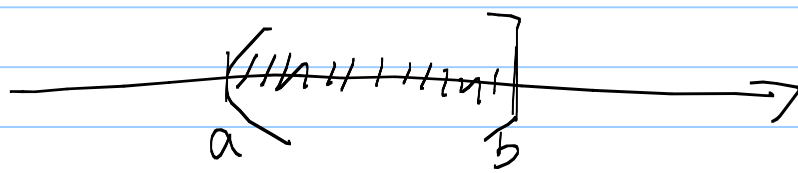
Q: $P(a \le X < b) = ?$

Ans: $P(X \le a) = F_X(a)$ by definition

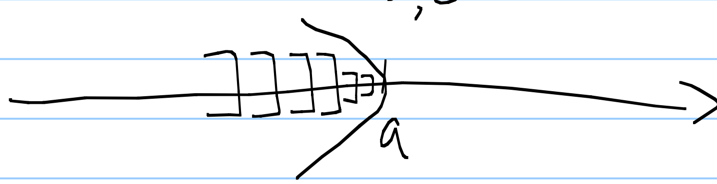
$$P(X>a) = 1 - F_X(a)$$



$$P(a<X\le b) = F_X(b) - F_X(a)$$

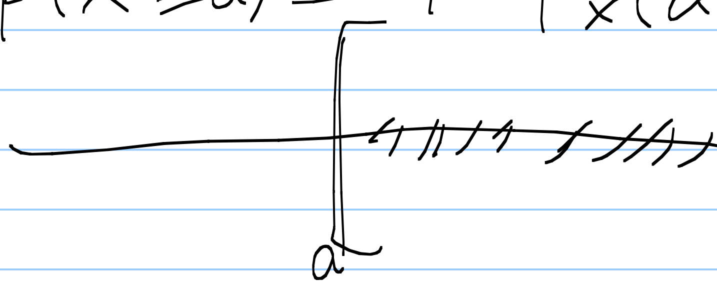


$$P(X<a) = \lim_{\varepsilon>0,\, \varepsilon\to 0} F_X(a-\varepsilon) \triangleq F_X(a^-)$$



Rewrite it as

$$P(X \ge a) = 1 - F_X(a^-)$$



$$P(a \le X \le b) = F_X(b) - F_X(a^-)$$

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$P(X=a) = P(a \le X \le a) = F_X(a) - F_X(a^-)$

the jump

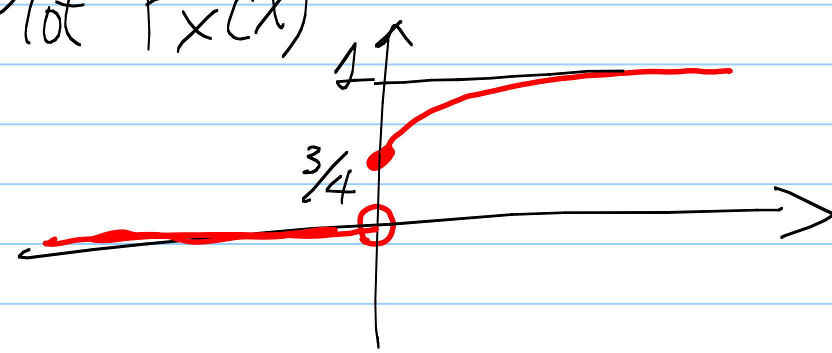
$P(a<X<b) = F_X(b^-) - F_X(a)$

$P(a \le X<b) = F_X(b^-) - F_X(a^-)$

Ex: HW6Q9 Prob 4.13

$$F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-\frac{1}{4}e^{-2x} & \text{if } 0 \le x \end{cases}$$

Plot $F_X(x)$



Q: $P(X \le 2)$, $P(X=0)$, $P(X<0)$,

$P(2<X<6)$, $P(X>10)$.

Ans: $F_X(2) = 1-\frac{1}{4}e^{-4}$

$F_X(0) - F_X(0^-) = (1-\frac{1}{4}e^{-0}) - 0 = \frac{3}{4}$

$F_X(0^-) = 0$

$F_X(6^-) - F_X(2) = (1-\frac{1}{4}e^{-12}) - (1-\frac{1}{4}e^{-4})$

$= \frac{1}{4}(e^{-4} - e^{-12})$

$$1-F_X(10) = 1-\left(1-\frac{1}{4}e^{-20}\right)$$
$$= \frac{1}{4}e^{-20} \quad \#$$

Q $P(0 < X \leq 6) = ?$

Ans: $F_X(6) - F_X(0) = \left(1-\frac{1}{4}e^{-12}\right) - \left(\frac{3}{4}\right) = \frac{1}{4}(1-e^{-12})$

Advantage of using cdf:

(3) Characterizing R.V.s of mixed type.

Discrete R.V. — $F_X(x)$ looks like Staircase, (containing only jumps)

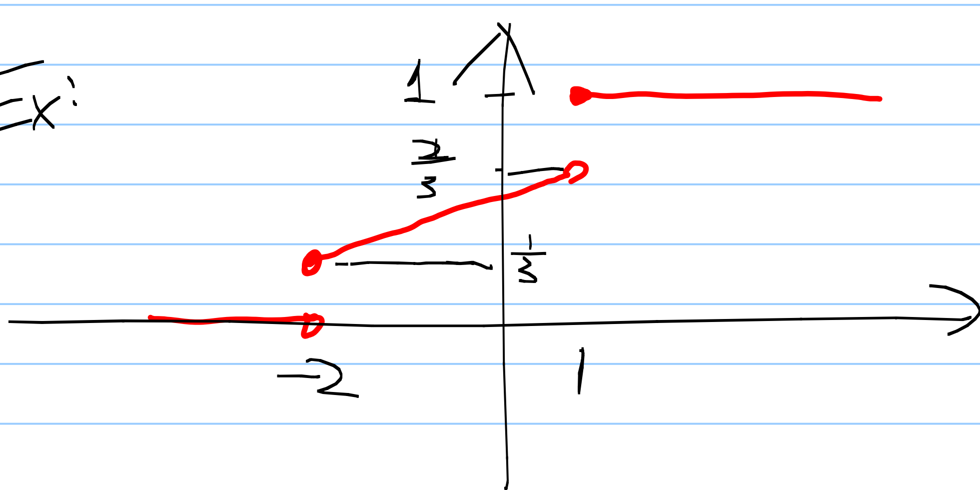
Conti — $F_X(x)$ is continuously increasing.

R.V. of mixed type

$F_X(x)$ contains some jumps & some continuously rising regions.

Ex: HW 6 Q9 Prob. 4.13.

Ex:



Ex: A real number X is chosen as follows: Flip a fair coin, if head, $X = \frac{1}{2}$

if tail: use a computer to pick randomly from $(0,1)$

Q: $F_X(x) = ?$

Ans: $F_X(x) = P(X \le x)$

$$= P(X \le x \text{ and head}) + P(X \le x \text{ and tail})$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ 0 + \frac{1}{2}\int_0^x 1\,dx = \frac{1}{2}x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} \times 1 + \frac{1}{2}\int_0^x dx = \frac{1}{2}x + \frac{1}{2} & \text{if } \frac{1}{2} \le x < 1 \\ \frac{1}{2} \times 1 + \frac{1}{2} \times 1 & \text{if } 1 \le x \end{cases}$$