$$P(X \le 6) = \sum_{k=0}^{6} P_k$$

$$= \sum_{k=0}^{6} \frac{\left(\frac{27}{2}\right)^k}{k!} e^{-\frac{27}{2}} , \quad 7 \text{ terms}$$

= 0.01925362

Q: $E(X) = ?$

Ans: $$E(X) = \sum_{k=0}^{\infty} P_k \; k$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{27}{2}\right)^k}{k!} e^{-\frac{27}{2}} \times k$$

$$= \sum_{k=1}^{\infty} \frac{\left(\frac{27}{2}\right)^k}{(k-1)!} e^{-\frac{27}{2}}$$

Let $k' = k-1$

$$= \left(\frac{27}{2}\right) \underbrace{\sum_{k'=0}^{\infty} \frac{\left(\frac{27}{2}\right)^{k'}}{(k')!} e^{-\frac{27}{2}}}_{\text{Total prob} = 1}$$

$$= \frac{27}{2} = \alpha$$

Q: $E(X(X-1)) = ?$

Ans: $$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} (k \cdot (k-1))$$

$$= \sum_{k=2}^{\infty} \frac{\alpha^k}{(k-2)!} e^{-\alpha}$$

$$= \alpha^2 \sum_{k=2}^{\infty} \frac{\alpha^{k-2}}{(k-2)!} e^{-\alpha}$$

let $k' = k-2$

$$= \alpha^2 \underbrace{\sum_{k'=0}^{\infty} \frac{\alpha^{k'}}{k'} e^{-\alpha}}_{=1}$$

$$= \alpha^2$$

Q: $E(X^2) = ?$ $\quad Var(X) = ?$

Ans: $X^2 = X(X-1) + X$

$$\Rightarrow E(X^2) = E(X(X-1)) + E(X)$$

$$= \alpha^2 + \alpha$$

$$Var(X) = E(X^2) - m^2 = \alpha^2 + \alpha - (\alpha)^2$$

$$= \alpha \quad \#$$

HW5 Q8

The number of page requests that arrive at a Web server is Poisson w. avg 6000 requests per minute.

Q: P(No request in 100 ms)

Q: P(5 to 10 requests in 100 ms)

Q: If more than 15 requests in 100 ms. The server crashes

P(server crashes) = ?

|| avg # of requests / 100 ms = 10

Ans: X is the number of packets in 100 ms.

X is Poisson with

$\alpha = 6000 \times \frac{1}{60} \times 0.1 = 10$ (reques per 100 msec)

$$P(X=0) = \frac{\alpha^0}{0!} e^{-\alpha} = e^{-10} \approx 4.54 \times 10^{-5}$$

$$A: P(5 \le X \le 10) = \sum_{k=5}^{10} \frac{10^k}{k!} e^{-10}$$

$$= 55.4\%$$

$$A: P(\text{crashes}) = P(X > 15) = 1 - P(X \le 15)$$

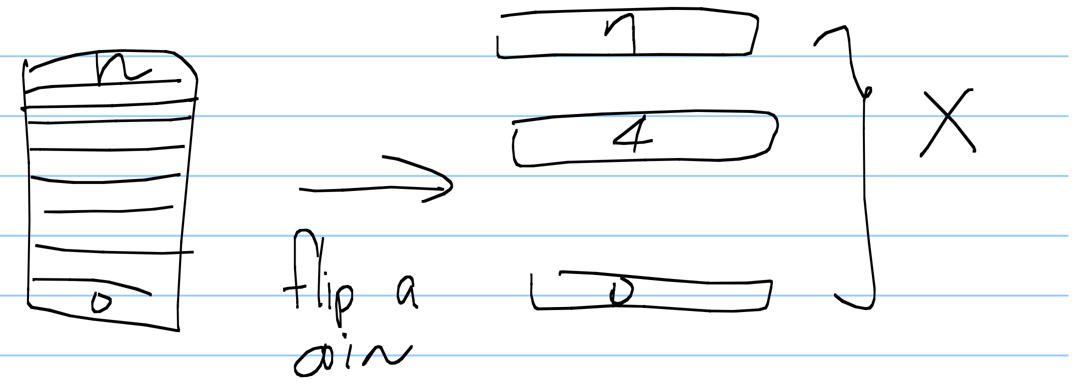
$$= 1 - \sum_{k=0}^{15} \frac{10^k}{k!} e^{-10}$$

$$\approx 4.87\%$$

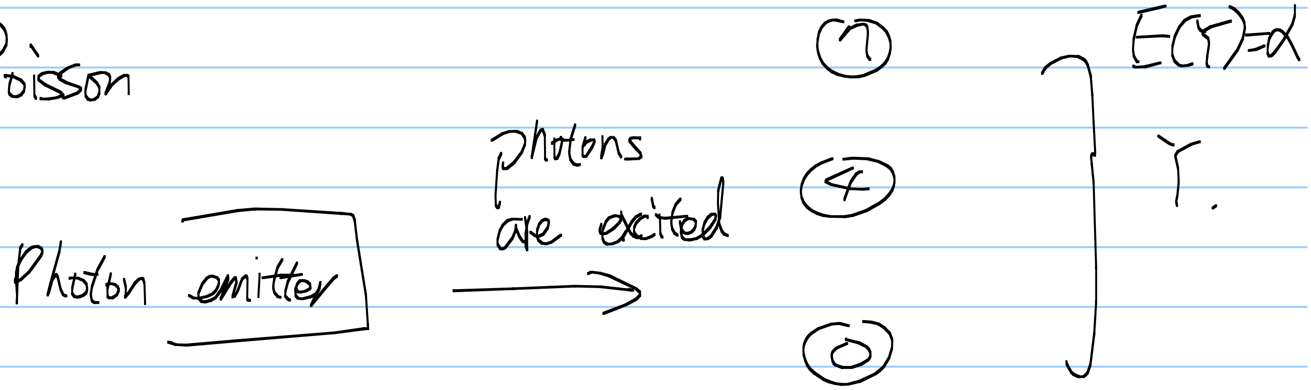
* The connection between binomial & Poisson distributions.

$E(X) = np$

Binomial



Poisson



The difference is that there are thousands of millions different photon that may go through the laser, but only a very small fraction of them can go through.

I.e. For binomial distribution, we keep $E(X) = np = \alpha$. & let $n \to \infty$, $p = \frac{\alpha}{n}$ so that $E(X) = E(Y)$

Then we have

Binomial

$$P_k = \frac{n!}{k!\,(n-k)!} \cdot p^k (1-p)^{n-k}$$

$$= \frac{\overbrace{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}^{k \text{ of them}}}{k!} \cdot p^k \cdot (1-p)^n$$

$$\cdot \frac{1}{(1-p)^k}$$

$$\approx \frac{1}{k!} \cdot (np)^k \cdot \left(1 - \frac{\alpha}{n}\right)^n \cdot \frac{1}{1}$$

$$= \frac{\alpha^k}{k!} e^{-\alpha} = P_k \quad \text{Poisson.}$$

* In sum: Poisson is the limit of a binomial with $n \to \infty$, $p = \frac{\alpha}{n}$

Note Title 2/11/2011

1. Many different R.Vs. (discrete thus far)
2. The W.A.
3. Expectation & variance
4. New computation skills.
5. The same counting principle

Continuous R.Vs.

1. Sample space is continuous.
2. The W.A is specified by the prob density function $f_X(x)$

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

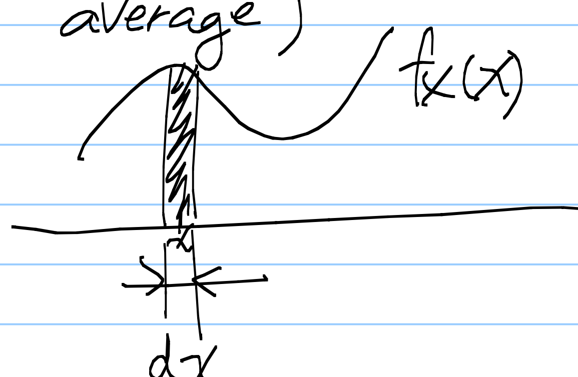
the area underneath $f_X(x)$.

Plotting $f_X(x)$ is just like plotting any function except that $f_X(x) \geq 0$.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

3. Expectation (Weighted average)

$$E(X) = \int_{-\infty}^{\infty} \underbrace{x}_{\text{face value}} \underbrace{f_X(x)\,dx}_{\text{Weight}}$$



$$E(X^2) = \int_{-\infty}^{\infty} \underbrace{x^2}_{\text{face value}} \underbrace{f_X(x)\,dx}_{\text{weight}}$$

$$E(e^X) = \int_{-\infty}^{\infty} e^x f_X(x)\,dx$$

Ex: $f_X(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

Q: Find $E(X) = ?$ Ans: $\int x f_X(x) = \int_0^3 x \frac{1}{3} dx = \frac{9}{6}$

* Expectation of a constant is the constant itself.

$$E(\pi) = \int_{-\infty}^{\infty} \pi f_X(x)\,dx = \pi \int_{-\infty}^{\infty} f_X(x)\,dx = \pi$$

* Expectation is linear

$$E(a\,g(X)) = a\,E(g(X))$$

$$\because \int_{-\infty}^{\infty} a g(x) f_X(x)\,dx = a \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$E(g_1(X) + g_2(X)) = E(g_1(X)) + E(g_2(X))$$

Again, we use the same formulas of expectation to define the "variance"

$$Var(X) \boxed{\overset{①}{=} E((X-m)^2)} = \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\,dx$$

$$\boxed{\overset{②}{=} E(X^2) - m^2}$$ However the computation is different

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - m^2$$

We can also define

* The n-th moment

$$E(X^n) = \int x^n f_X(x)\, dx$$

* The n-th central moment

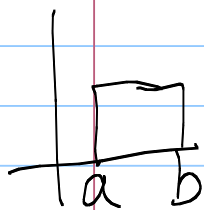
$$E((X-m)^n) = \int (x-m)^n f_X(x)\, dx$$

where m is the mean $E(X)$ #

0715

Important Conti R.V. Table 4.1 p.164

1. Uniform Random Var. with para $a < b$



$S_X = [a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

ex: $a = 2.7.$ $b = \pi$

Ex: The computer picks up a random number between $[a, b]$

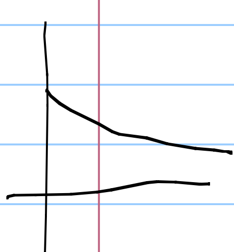
$$E(X) = \int x f_X(x) dx$$

$$= \int_a^b x \times \frac{1}{b-a} dx = \frac{a+b}{2}$$

$$Var(X) = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \times \frac{1}{b-a} dx$$

$$= \frac{(b-a)^2}{12} \quad \text{(exercise)}$$

2. Exponential R.V. with para $\lambda > 0$



$S = [0, \infty)$ any non-negative real number

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

λ can be π, 0.223, ...

Ex: Customers arrive at the average rate λ customer/per $\boxed{\text{unit time}}$. The amount of waiting time for the 1st customer is modeled by an exponential R.V.

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx$$

integration by part

$$= \frac{1}{\lambda} \quad \boxed{\text{unit time.}}$$

Ex: average $\lambda =$ 30 customers/ hour

$$E(X) = \frac{1}{\lambda} = \frac{1}{30} \ \underline{\underline{\text{hour}}}$$

unit time

We can also say
the average arrival is $\lambda = 0.5$ customers/ minute

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.5} = 2 \ \underline{\underline{\text{min.}}}$$

unit time

Q: P(The first time we see a customer is ≥ 30 min)

Ans: $P(X > 0.5)$ || $P(X > 30)$

$$= \int_{0.5}^{\infty} 30 \cdot e^{-30x}\,dx \quad \Big\| \quad \int_{30}^{\infty} 0.5e^{-0.5x}\,dx$$

*Bernoulli (p) : flip a single coin with head prob. p. [077]

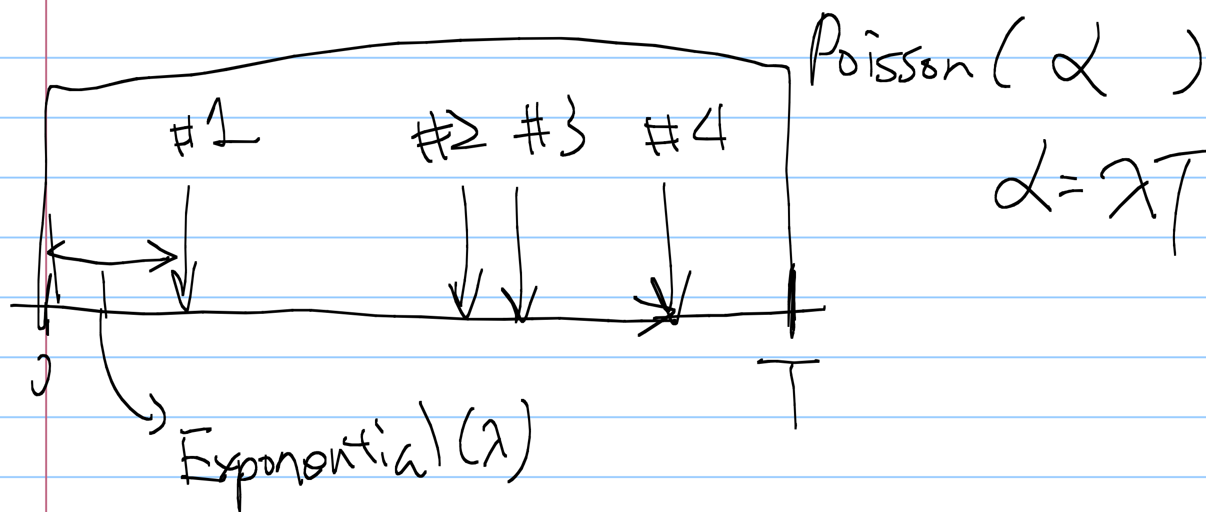
* Binomial (n, p) : flip n coins with head prob. p.
Count the total # of heads.

* Geometric (p) : flip a coin until "head".
Count the # of tails before the first head.

* Poisson (α) : For a given amount of time T, α is the expected # of arrivals in time T.
Count the actual # of arrivals.

* Poisson is a limiting case of binomial.

* Exponential (λ) : λ is the expected # of arrivals in a unit time.
Count the actual waiting time of the first arrival.



E.g.

P(waiting time > 4.5) = P(No arrival in 4.5 unit time)

$$\int_{4.5}^{\infty} \lambda e^{-\lambda x}\,dx = \frac{(4.5\lambda)^0}{0!} \cdot e^{-(4.5\lambda)}$$

A question that is similar to HW5Q7.

0729

Q: X is an exponential R.V w. para. λ

Show that for any $a, b > 0$.

$$P(X > a+b \mid X > a) = P(X > b)$$

Ans: RHS: $\int_b^\infty \lambda e^{-\lambda x} dx = e^{-b}$

LHS: $\dfrac{P(X > a+b)}{P(X > a)}$

$$= \frac{\int_{a+b}^\infty \lambda e^{-\lambda x} dx}{\int_a^\infty \lambda e^{-\lambda x} dx} = \frac{e^{-(a+b)}}{e^{-a}}$$

$$= e^{-b}$$

\# It is called the $\boxed{\text{memoryless property.}}$
Why? Recall that X generally models the time you wait for the first customer.

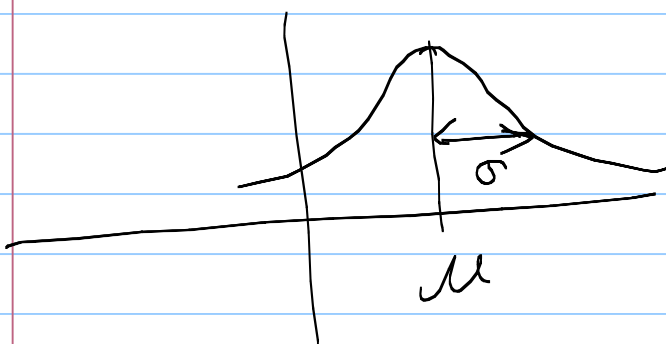
Given that I have waited for a secs, the prob that I have to wait for additional b secs does not depend on how large/small a is. There is no memory to how long I have waited.

L080

3. Gaussian / Normal R.V. w. para μ, σ.

$S = (-\infty, \infty)$ any real number

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \underbrace{\int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}$$

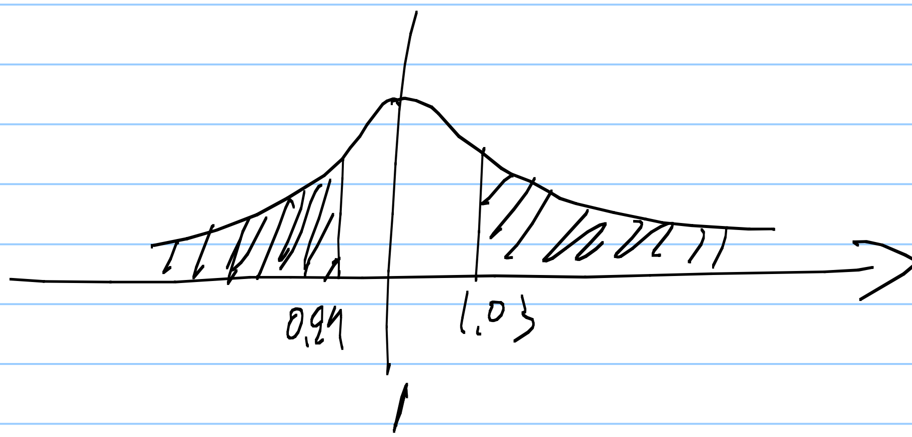
Odd function

$$+ \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= 0 + 1$$

$Var(X) = \sigma^2$ (hard to derive)

Ex: The reading of the GPS device X is a Gaussian R.V with $\mu = 1$ $\sigma = 0.01$
(where $\mu = 1$ is the actual location of the device



Q: What is the prob the GPS reading is 3% off the actual location.

Ans: $P(X < 0.97 \text{ or } X > 1.03)$

$$= \int_{-\infty}^{0.97} \frac{1}{\sqrt{2\pi \times 0.01^2}} e^{-\frac{(x-1)^2}{2 \times 0.01^2}} dx$$

$$+ \int_{1.03}^{+\infty} \frac{1}{\sqrt{2\pi \times 0.01^2}} e^{-\frac{(x-1)^2}{2 \times (0.01)^2}} dx$$

by MATLAB $\approx 0.27\%$

Other important conti R.Vs. include Laplacian, and Rayleigh R.Vs. See p.165 for their S and f_X description.

Discrete & Conti R.Vs are similar ex: $E(X+X^2)=E(X)+E(X^2)$

$$Var(X)=E((X-m)^2)=E(X^2)-m^2$$

How to use a unifying description to describe both types of R.V?

* Cumulative Distribution Function (cdf)

$\boxed{F_X(x)}$ where the input can be any real number ex: $x=0.01$, $x=\pi\times10^5$

$$\begin{tikzcd}
\text{Discrete R.V.} & & \text{Conti R.V} \\
\text{pmf } p_k \arrow[u] \arrow[r, shift left, "\mathrm{I.}"] & \text{cdf } F(x) \arrow[ul] \arrow[ur] \arrow[l, shift left, "\mathrm{II.}"] \arrow[r, shift right, "\mathrm{IV}"'] & \text{pdf } f_X(x) \arrow[u] \arrow[l, shift right, "\mathrm{III}"']
\end{tikzcd}$$

Definition

$$F_X(x) \triangleq P(X \le x).$$

$\boxed{\mathrm{I}}$ ※ Discrete R.V. from p_k to $F_X(x)$

Ex: X is a bernoulli R.V. with $p=\frac{1}{3}$

Find its cdf $F_X(x)$, Plot $F_X(x)$

Ans: $F_X(x) = P(X \le x)$ (Note that X is integer but x does not need to be integer)

$$F_X(\pi) = P(X \le \pi)$$

$$= \frac{2}{3} + \frac{1}{3} = 1.$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2/3 = p_0 & 0 \le x < 1 \\ \frac{2}{3} + \frac{1}{3} = p_0 + p_1 & 1 \le x \end{cases}$$